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Représentations Effectives en Géométrie Algébrique Réelle et Optimisation Polynomiale

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Effective Representations in Real Algebraic Geometry and Polynomial Optimization

Lorenzo BALDI

Jury

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Abstract

In polynomial optimization, two different and dual approaches are considered: the approximation of positive polynomials using sums of squares (SoS), that translates into Lasserre's SoS hierarchy, and the approximation of measures using truncated positive linear functionals (or truncated pseudo-moment sequences), that translates into Lasserre's moment hierarchy. In this thesis, we investigate exact and approximate representation properties in both cases.

The representation of positive polynomials in terms of sums of squares is a central question in real algebraic geometry, that is answered by the Positivstellensätze. In particular, we investigate effective version of Putinar's Positivstellensatz, and provide new bounds on the degree of representation of a strictly positive polynomial on a basic compact semialgebraic set S, under the Archimedean condition. These bounds involve a parameter ε, measuring how close is the strictly positive polynomial to have a zero on the semialgebraic set: these are the first bounds with a polynomial dependency on ε^{-1}. The bounds also show a new explicit dependency on the Łojasiewicz exponent $Ł$ and constant $\mathfrak{c}$, arising from a Łojasiewicz inequality between the distance and semialgebraic distance functions from S. We analyze in detail regular cases, where we can show that the Łojasiewicz exponent is equal to one and we have explicit bounds for the Łojasiewicz constant.

We interpret our effective Putinar's Positivstellensatz as a result of quantitative approximation of positive polynomials, and deduce the first general polynomial convergence for Lasserre's hierarchies. On the dual side, we deduce convergence rates of truncated positive linear functionals (or truncated pseudo-moment sequences) to measures.

We then move to exact representations on the dual side. We investigate properties of the dual cones of truncated quadratic modules, and we introduce the concept of exactness for Lasserre's moment hierarchy, that is closely related to the flat truncation property. We show that the dual of the moment hierarchy coincides with an extended SoS hierarchy, and we detail the analysis for zero dimensional semialgebraic sets. Finally, we apply the obtained results to the study of flat truncation. We give the first necessary and sufficient condition for flat truncation, under the finite convergence assumption, giving bounds for the order of relaxation needed. As corollaries, we conclude that flat truncation holds under generic assumptions, and we give a unified presentation of different results in the zero dimensional case. Finally, we briefly discuss examples of the alternate current - optimal power flow problem.

As an application, we present a new algorithm for computing the real radical of an ideal I. We exploit properties of truncated positive linear functionals and techniques from numerical algebraic geometry. We give an effective, general stopping criterion on the degree, to detect when the kernel of the moment matrix of a generic linear functional can be used to compute equations for the irreducible components of the real variety defined by I. Finally, we compute the real radical as the intersection of real prime ideals lying over I, and illustrate this approach in several examples.

Key words: Moments, Positive Polynomials, Duality, Real Algebraic Geometry, Optimization

Resume

Dans le domaine de l'optimisation polynomiale, deux approches différentes et duales sont considérées : l'approximation de polynômes positifs à l'aide de sommes de carrés (SoS), qui se traduit par la hiérarchie SoS de Lasserre, et l'approximation de mesures à l'aide de fonctions linéaires positives tronquées (ou de séquences de pseudo-moments tronquées), qui se traduit par la hiérarchie des moments de Lasserre. Dans cette thèse, nous étudions les propriétés de représentation exactes et approchées dans les deux cas.

La représentation des polynômes positifs en termes de sommes de carrés est une question centrale en géométrie algébrique réelle, à laquelle répondent les Positivstellensätze. En particulier, nous étudions une version effective du Positivstellensatz de Putinar, et fournissons de nouvelles bornes sur le degré de représentation d'un polynôme strictement positif sur un ensemble semialgébrique de base S compact, sous la condition Archimedienne. Ces bornes font intervenir un paramètre ε, qui mesure a quelle distance se trouve le polynôme strictement positif d'avoir un zéro sur l'ensemble semialgébrique: ce sont les premières bornes avec une dépendance polynomiale de ε^{-1}. Dans les bornes, on trouve également une nouvelle dépendance explicite de l'exposant de Łojasiewicz $Ł$ et de la constante $\mathfrak{c}$, provenant d'une inégalité Łojasiewicz entre les fonctions de distance et de distance semialgébrique de S. Nous analysons en détail les cas réguliers, dans lesquels nous pouvons montrer que l'exposant Łojasiewicz est égal à un et nous avons des limites explicites pour la constante Łojasiewicz.

Nous interprétons notre Positivstellensatz effectif de Putinar comme un résultat d'approximation quantitatif des polynômes positifs, et déduisons la première convergence polynomiale générale pour les hiérarchies de Lasserre. Du point de vue dual, nous déduisons les taux de convergence des fonctions linéaires positives tronquées (ou de séquences de pseudo-moments tronquées) vers les mesures.

Nous passons ensuite aux représentations exactes dans le dual. Nous étudions les propriétés des cônes duaux des modules quadratiques tronqués, et nous introduisons le concept d'exactitude pour la hiérarchie des moments de Lasserre, qui est étroitement lié à la propriété de troncation plate. Nous montrons que le dual de la hiérarchie des moments coïncide avec une hiérarchie SoS étendue, et nous détaillons l'analyse pour les ensembles semialgébriques de dimension zéro. Enfin, nous appliquons les résultats obtenus à l'étude de la troncation plate. Nous donnons la première condition nécessaire et suffisante pour la troncature plate, sous l'hypothèse de convergence finie, en donnant des limites pour l'ordre de relaxation nécessaire. Comme corollaires, nous concluons que la troncation plate est verifiée sous des hypothèses génériques, et nous donnons une présentation unifiée des différents résultats dans le cas de la dimension zéro.

Comme application, nous présentons un nouvel algorithme pour calculer le radical réel d'un idéal I. Nous exploitons les propriétés des fonctions linéaires positives tronquées et les techniques de la géométrie algébrique numérique. Nous donnons un critère d'arrêt efficace et général sur le degré, pour détecter quand le noyau de la matrice des moments d'une fonction

linéaire générique peut être utilisé pour calculer les équations des composantes irréductibles de la variété réelle définie par I. Enfin, nous calculons le radical réel comme l'intersection d'idéaux premiers réels contenant sur I, et illustrons cette approche par plusieurs exemples.
Mots-clés: Moments, Polynômes Positifs, Dualité, Géométrie Algébrique Réelle, Optimisation

In memory of Carlo Casolo and David Ciampi

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Contents

Introduction

A ubiquitous pattern in mathematics has always been the search for various representations of a given object. Indeed, different representations can shade light to distinct properties, or be useful for different generalizations, or even provide a better encoding of the object for different computations.

In this thesis, we study exact and approximate representations of real polynomials (in particular *positive polynomials*), and the associated dual objects.

Without any doubt, it is impossible to mention all the possible representations of polynomials that are of interest, and their infinite applications in algebra and geometry. To name only a few of them: the factorization, i.e. the representation as a product of irreducible polynomials; the Waring decomposition, i.e. the representation as sum of powers of linear forms; or Artin's solution to Hilbert's seventeenth problem, i.e. the representation of globally nonnegative polynomials as ratios of *sums of squares*. On a different perspective, polynomials can also be used to represent other objects. For instance, the Stone-Weierstrass theorem says that we can approximately represent continuous functions on compact sets using polynomials, with any prescribed accuracy.

Considering dual objects, one can naturally look at representations of linear functionals acting on polynomials. One of the most natural classes of such linear functionals are those *induced by measures*, that is linear functionals that coincide with the integration with respect to a measure. Problems regarding existence, uniqueness and properties of the representing measures define what is called the *moment problem*. This subject has also a long history and infinite applications in pure and applied mathematics. To name another possible representation of linear functionals acting on polynomials, recall Macaulay's theory of inverse systems, representing linear functionals vanishing on a (zero dimensional) ideal.

At the beginning of the new millennium, a new research topic born, as an application of representation properties of positive polynomials and linear functionals: *Polynomial Optimization*. Its rich structure and vast application possibilities posed naturally many interesting questions, that captured the interest of a large and various research community in the past two decades.

This thesis follows this point of view: guided by questions arising from polynomial optimization, we investigate exact and approximate representations of positive polynomials and linear functionals.

Context of the thesis

The problem of representing positive[1] polynomials has long history. A natural class of globally positive polynomials are those that can be written as sums of squares of other

[1]We follow the French tradition, and call a function f *positive* on a domain D if $f \geq 0$ on D and *strictly positive* on D if $f > 0$ on D.

polynomials. This representation can be seen as a *certificate* of positivity, and a question arises naturally: can we find such a representation for *all* the globally positive polynomials? This problem was studied by D. Hilbert [Hil88], that gave a negative answer (except for small number of variables or low degree polynomials). Then, Hilbert asked in his famous seventeenth problem if a generalized representation was always possible, namely if it is possible to represent all globally positive polynomials using *ratios* of sums of squares of polynomials. A positive answer to this question was given by E. Artin [Art27], using the celebrated theory of real closed fields. Several years later, this rational representation was generalized to arbitrary semialgebraic sets (i.e. sets defined by a boolean combination of sign conditions involving a finite number of polynomials) by J. L. Krivine [Kri64] and rediscovered by G. Stengle [Ste74].

All the representation theorems above involve a denominator in the representation. It was then a great achievement when K. Schmüdgen [Sch91] proved the first general denominator free representation for strictly positive polynomials on compact *basic* semialgebraic sets $S = \mathcal{S}(\mathbf{g})$ (that is, sets defined by finitely many non-strict polynomial inequalities $\mathbf{g} = g_1, \dots g_r$). In particular, Schmüdgen proved a representation of the strictly positive polynomial as an element of the *preordering* $O = \mathcal{O}(\mathbf{g})$ generated by the defining polynomial inequalities, i.e. a representation as weighted sum of products of the defining polynomial inequalities, with sum of squares coefficients. This result is known as *Schmüdgen's Positivstellensatz*. The representation was obtained solving the dual moment problem, that is showing that any linear functional on the polynomial ring that is positive on the preordering O is induced by a measure supported on the compact semialgebraic set, and the proof combined the result of Krivine and Stengle with functional analysis techniques. One drawback on this representation is that the number of addenda is exponential in the number of defining inequalities. This problem was solved by M. Putinar [Put93]: replacing compactness with the slightly stronger Archimedean condition, he showed the existence of a representation for strictly positive polynomials as elements of the *quadratic module* $Q = \mathcal{Q}(\mathbf{g})$ generated by the polynomial inequalities, i.e. a representation as weighted sum of the defining inequalities, with sum of squares coefficients. Therefore, the number of addenda needed in this representation is *linear* in the number of defining inequalities. This result is known as *Putinar's Positivstellensatz*.

We have seen above how representations of positive polynomials using sum of squares has a long story, and this is of interest because the sum of square polynomials provide explicit certificates of positivity. Then, given a positive polynomial one would like to have *algorithms* to produce such a certificate of positivity, and the first step is naturally the investigation of sum of squares polynomials from this point of view. On this direction, of primary importance have been the works of N. Z. Shor [Sho87] and M. D. Choi, T. Y. Lam, and B. Reznick [CLR95], identifying the link between sum of squares, Gram matrices and convexity.

The ground was finally ready to reveal the connection between sum of squares representations and semidefinite programming. This connection was developed independently by J. B. Lasserre [Las00; Las01] and P. Parrilo [Par00; Par03], and provided two dual hierarchies of convex cones, the *moment hierarchy* and the *sum of squares hierarchy*, that are defined by semidefinite constraints and can be used to compute lower bounds for the global minimum of an objective polynomial function f on a basic closed semialgebraic set S. These hierarchies, indexed by a natural number d, produce two sequences of lower bounds that converge, under the Archimedean assumption, to the minimum f^* of the objective function. For the sum of squares hierarchy, we consider at order d the polynomials in the quadratic module generated in degree $\leq 2d$ (i.e. we bound the degree of the sum of square coefficients), while for the moment hierarchy we consider the dual cone, namely the linear functionals that are positive, when applied to polynomials in the quadratic module that are generated in degree $\leq 2d$. Convergence of these hierarchies can be deduced from Putinar's Positivstellensatz, or from Schmüdgen Positivstellensatz when we consider also the products of the defining inequalities.

One natural question is then to investigate *convergence rates* of these hierarchies. The problem is easily seen to be equivalent to the following questions: given a strictly positive polynomial f on a compact basic semialgebraic set S, what is the degree needed for the sum of squares coefficients of a representation of f as an element of the preordering O? Or equivalently, under the Archimedean condition: what is the degree needed for the sum of squares coefficients of a representation of f as an element of the quadratic Q? The answer to these questions does not depend only on the degree or the norm of f, but also on the minimum $f^* > 0$ of f on S. We refer to this problem as the *Effective* Schmüdgen's and Putinar's Positivstellensatz. The first general version of the Effective Schmüdgen's Positivstellensatz was proven by M. Schweighofer [Sch04], and the Effective Putinar's Positivstellensatz was successively investigated by J. Nie ans M. Schweighofer [NS07]. Proving this kind of results is difficult: for instance, only recently H. Lombardi, D. Perrucci, and M.-F. Roy [LPR20] proved a general effective version of Krivine and Stengle's theorem. The bounds for the Effective Schmüdgen's Positivstellensatz and the Effective Putinar's Positivstellensatz have an important difference: the Effective Schmüdgen's Positivstellensatz in [Sch04] has a polynomial dependence on f^{*-1}, while in the Effective Putinar's Positivstellensatz in [NS07] the dependence is *exponential* on f^{*-1}. This exponential dependency leads to a logarithmic convergence rate for the sum of squares and moment lower approximations based on Putinar's Positivstellensatz to f^*. Naturally it was asked whether it is possible or not to obtain a polynomial dependency on f^* for the Putinar's Positivstellensatz as well.

But despite the theoretical slow convergence rate, both in simple toy examples and in real world problems *finite convergence* of the hierarchies was observed, i.e. that a finite order d of the hierarchy we have equality between the lower approximation and the minimum f^*. For the sum of squares hierarchy, this property is implied by representations of positive polynomials *with zeros* on S as elements of the quadratic module Q (or of the preordering O). The existence of this kind of representations was investigated by C. Scheiderer [Sch03; Sch05a] for polynomials with isolated zeros, and a particularly useful case was studied by M. Marshall [Mar06], where he shows the existence of the representation in Q under the so-called *Boundary Hessian Conditions* at the isolated zeros of f. J. Nie [Nie14] have finally shown that these conditions are *generic* (in the Archimedean case), implying generic finite convergence for the sum of squares and the moment hierarchies.

However, in practice one cannot detect finite convergence using the sum of squares representation, but the moment hierarchy has the necessary properties. Indeed, using a rank condition on the moment matrix of an optimal, feasible positive linear functional (or, equivalently, of an optimal feasible pseudo-moment sequence) one can certify the finite convergence of the moment hierarchy. This criterion is based on the solution of the truncated moment problem by R. Curto and L. Fialkow [CF96; CF00], the so-called *flat extension* criterion. This criterion is effective in practice, and under some assumptions has been proven to be equivalent to the finite convergence of the sum of squares hierarchy by J. Nie [Nie13b], but a complete theoretical understanding of the criterion in polynomial optimization is missing. For instance, there are no necessary and sufficient conditions to determine if we can certify finite convergence using flat truncation, and it is not clear also if the generic conditions that imply finite convergence, imply flat truncation as well.

Finally, let us mention an application of these techniques to the solution of systems of real polynomial equations $\mathbf{f}$. The real varieties X defined by these systems are particular and interesting cases of semialgebraic sets, and solving the polynomial system (over the reals) means to find the simplest equations defining the (real) vanishing ideal of X. In the case when X is zero dimensional, solving the system means equivalently finding all the points of X. One can use moment relaxations (with no objective function) to find these equations, and in the zero dimensional case the flat extension criterion certificates that the correct equations have been computed. These results were discovered by J.B. Lasserre, M. Laurent and P. Rostalski [LLR08] and the computation improved by J.B. Lasserre, M. Laurent, B. Mourrain, P. Rostalski and P. Trébuchet [Las+13]. Their algorithms apply only in the zero dimensional case, while the positive dimensional case remains open.

In this section, we have given a brief historical account of results related to this thesis, namely for representations of positive polynomials, the moment problem and polynomial optimization. However, we have only touched the surface of these subjects, omitted many theoretical contributions and almost ignored their important real life applications. We provide more precise and detailed references in the introduction to every chapter, and refer to the books [BCR98; PD01; BPR06; Mar08; Lau09; Las10; BPT12; Las15; Sch17; HKL20; Pow21] for more complete discussions of these topics.

Contributions

In the thesis, we answer some open questions that have been raised in the previous section. Hereafter we summarize the main results of the thesis, and refer to the introduction to every chapter for more details and for the related literature.

In **Chapter 2**, we investigate the problem of representation of strictly positive polynomial on compact basic semialgebraic set S, under the Archimedean condition, as elements of the quadratic module defining S. The result ensuring the existence on such a representation is the celebrated *Putinar's Positivstellensatz*.

In particular, we produce an *effective* version of Putinar' Positivstellensatz, i.e. we determine degree bounds for the sum of squares coefficients in such a representation. The main result of the chapter is the following.

Theorem 2.2.14. *Assume $n \geq 2$ and let $g_1, \ldots, g_r \in \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \ldots, x_n]$ satisfying the normalization assumption* (2.1). *Let $f \in \mathbb{R}[\mathbf{x}]$ such that $f^* = \min_{x \in S} f(x) > 0$. Let $\mathfrak{c}$, $Ł$ be the Łojasiewicz coefficient and exponent given by Definition 2.2.8. Then $f \in \mathcal{Q}_\ell(\mathbf{g})$ if*

$$\begin{aligned} \ell &\geq O(n^3 2^{5nŁ} r^n \mathfrak{c}^{2n} d(\mathbf{g})^n d(f)^{3.5nŁ} \varepsilon(f)^{-2.5nŁ}) \\ &= \gamma(n, \mathbf{g})\, d(f)^{3.5nŁ} \varepsilon(f)^{-2.5nŁ}, \end{aligned}$$

where $\gamma(n, \mathbf{g}) \geq 1$ depends only on n and $\mathbf{g}$.

Let us briefly explain the result, to clarify our contribution compared to the existing literature. In the theorem, the most important parameter is $\varepsilon(f) = f^*/\|f\|$, where f^* denotes the minimum of f on $S = \mathcal{S}(\mathbf{g})$ and $\|f\|$ is the max norm of f on $[-1,1]^n$. $\varepsilon(f)$ is therefore a measure of how close is f to have a zero on S. The key point of the theorem is the *polynomial* dependency of the degree ℓ on $\varepsilon(f)^{-1}$. Indeed, the only general result previously available in the literature [NS07] had an *exponential* dependency on $\varepsilon(f)$ [2], and it was an open question since the result in [NS07] if a polynomial dependency like the one exhibited by Theorem 2.2.14 was possible or not.

Another improvement in our result compared to the result of Nie and Schweighofer is the explicitness of the bounds. Indeed, all the parameters of the bound in Theorem 2.2.14 are clearly described. On the contrary, the exponent of $\varepsilon(f)^{-1}$ in [NS07] is not explicit.

In particular, the exponent of $\varepsilon(f)^{-1}$ in our result is the *Łojasiewicz* exponent $Ł$, arising from a Łojasiewicz inequality involved in the proof of the result, see Definition 2.2.8. This inequality is also present in the proof of Nie and Schweighofer, but its role has been clarified and this inequality analyzed in more details. Indeed, we were able to prove that the Łojasiewicz exponent is equal to one under regularity condition, usually assumed in polynomial optimization problems. This result is presented in Theorem 2.3.9 and Theorem 2.3.13, where we furthermore give estimates for the Łojasiewicz constant in terms of geometric properties of the defining inequalities. To the best of our knowledge, this is the first analysis in the literature of the Łojasiewicz exponent in regular cases and the first estimate for the Łojasiewicz constant.

The motivation and the importance of the effective Putinar's Positivstellensatz is coming from polynomial optimization. Indeed, one can determine convergence rates for the Lasserre's sum of squares and moment hierarchies using this result. While an exponential dependence on $\varepsilon(f)^{-1}$ leads to a *logarithmic* convergence of the lower bound to the minimum, the polynomial dependency on $\varepsilon(f)^{-1}$ in Theorem 2.2.14 gives a *polynomial* convergence of the lower bounds. This general result was not yet proven in the literature, and it is presented in Theorem 2.4.3.

Theorem 2.2.14 gives also a quantitative inner approximation of positive polynomials using polynomials in the quadratic module, with degree bounds. This result is described in Theorem 2.4.1.

On the dual side, the dual cones of truncated quadratic modules are outer approximations of the cones of measures, and a section of this dual cone defines the feasible positive linear functionals for the moment hierarchy. In Theorem 2.5.9, we deduce the first general convergence rates of these outer approximations to moments of probability measures.

In **Chapter 3** we study exact representations of truncated positive linear functionals that are feasible for moment relaxations. In particular, we focus on *minimizing* linear functionals, i.e. those functionals that applied to the objective function f give the moment lower approximation of the minimum $f^*_{\mathrm{Mom},d}$.

We introduce the concept of *exactness* of the moment hierarchy, in order to study the outer approximation of the measures supported at the minimizers of f on S. We highlight connections with the flat truncation property, used to certify finite convergence of the moment hierarchy.

The main result of the chapter is the following theorem.

[2] The $\varepsilon(f)$ used by Nie and Schweighofer differs for the one of Theorem 2.2.14 for the choice of the norm, that is a weighted norm on the coefficients, while our is the max norm on $[-1,1]^n$. However, these norms are equivalent, and this makes the comparison, up to a constant, possible. See the introduction to Chapter 2 for more details.

Theorem 3.5.4. *Assume that we have moment finite convergence. Then* $\dim \frac{\mathbb{R}[\mathbf{x}]}{\operatorname{supp}(Q+(f-f^*))} = 0$ *if and only if there exists d such that a generic* $\Lambda^* \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ *has flat truncation.*

In particular, if $\rho = \rho(S^{\min})$, $D = \max(d_{\mathbf{g}}, \lceil \frac{\deg(f)}{2} \rceil)$ *and* $\delta \in \mathbb{N}$ *is such that* $f - f^* \in \overline{\mathcal{Q}_{2\delta}(\mathbf{g})}$, *flat truncation happens for* $\Lambda^* \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ *generic at degree* $\rho - 1$ *when d is such that:*

(i) $(\sqrt[\mathbb{R}]{\operatorname{supp} Q(\mathbf{g})})_{2\delta+2\rho+2D-\deg(f)-2} \subset \overline{\mathcal{Q}_{2d}(\mathbf{g})}$;

(ii) $\mathcal{I}(S^{\min})_{2\rho+2D-2} \subset \overline{\mathcal{Q}_{2d}(\mathbf{g}) + (f-f^*)_{2d}}$;

(iii) $\delta + 2\rho + 2D - \deg(f) - 2 \leq d$.

Although extensively used in polynomial optimization, the flat truncation (or flat extension) property is not completely understood theoretically in this context. In the theorem, we provide the first necessary and sufficient condition for the flat truncation property, and give the first degree bounds for the order of the relaxation needed to achieve flat truncation.

The proof of the theorem requires a detailed analysis of the dual cones of (truncated) quadratic modules, and in particular we show that the moment hierarchy coincides with an extended sum of squares hierarchy in Theorem 3.4.3 and Theorem 3.4.11. This result is also motivated from the duality theory in conic programming.

The analysis required for the proof of the theorem gives a detailed understanding of the duality between Lasserre's moment and sum of squares hierarchies, and it allows to create several new examples and counterexamples for finite convergence and exactness properties of the sum of squares and moment hierarchies. For instance, we describe an optimization problem on a finite semialgebraic set with finite converge of the hierarchies, but whose convergence cannot be certified using flat truncation (see Example 3.3.12).

A key ingredient for the proof of Theorem 3.5.4 is the analysis of the zero dimensional case. Theorem 3.4.19 and Theorem 3.4.20 give a complete description of the correspondence between zero dimensionality and flat truncation, generalizing existing results for finite real varieties and preorderings defining zero dimensional semialgebraic sets.

As consequences of Theorem 3.5.4, we show that flat truncation holds under generic regularity properties (Theorem 3.5.7), and apply the result to finite semialgebraic sets and polar ideals (Theorem 3.5.15).

In **Chapter 4** we use generic positive linear functionals to compute solutions of real polynomial systems, i.e. equations for the *real radical* of the polynomials. While the zero dimensional case is established, with the flat truncation criterion that can be used to certify that the kernel of the moment matrix generates the real radical, the positive case is much more challenging. We give a new, short proof in Theorem 4.2.1 a new, short proof that the equations for the real radical can be computed from the kernel also in the positive dimensional case. We then search for stopping criterions to determine if these equations have been computed using a moment matrix of a given order.

The final algorithm, which is the main result of the chapter, is Algorithm 4.5.1. We describe the steps of the algorithm and highlight the main contributions of the chapter the meantime. After computing a generic truncated positive linear functional Λ^*, using semidefinite programming and an interior point solver (step (ii)), we compute a basis of $\operatorname{Ann}_d(\Lambda^*)$. The computation (step (iii)) is done through Algorithm 4.3.1, which improves previous algorithms exploiting properties of positive linear functionals. We then compute a numerical irreducible decomposition of the complex associated variety, and then check if these varieties are totally

Algorithm 4.4.1: Real radical

Input: Polynomials $\mathbf{f} = (f_1, \ldots, f_s) \subset \mathbb{R}[\mathbf{x}]$.
$d := \max(\deg(\mathbf{f}_i), i = 1, \ldots, s) - 1$; success := false;
Repeat until success

(i) $d := d + 1$

(ii) Compute a generic element Λ^* of $\mathcal{L}_{2d+2}(\pm \mathbf{f})$

(iii) Compute a graded basis $\mathbf{k}$ of $\mathrm{Ann}_d(\Lambda^*)$ (Algorithm 4.3.1)

(iv) Compute the numerical irreducible components X_i of $V_{\mathbb{C}}(\mathbf{k})$ (described by witness sets)

(v) For each component X_i, check that X_i is real (Algorithm 4.4.1). If not repeat from step (i).

(vi) Set success := true

(vii) For each component X_i compute defining equations $\mathbf{h}_i = \{h_{i,1}, \ldots, h_{i,n+1}\}$ of X_i

Output: The polynomials $\mathbf{h}_i$ generating the minimal real prime ideals $\mathfrak{p}_i$ lying over $(\mathbf{f})$.

real. This check (step (v)) is done with a new algorithm (Algorithm 4.4.1), that reduces the problem to the hypersurface case (using a generic real projection), and then checks if the sign changing criterion is satisfied solving a polynomial optimization problem. The correctness of this step follows from Theorem 4.4.10. If every component is totally real, we compute the (real) equations of these components, that are the minimal real primes lying over the initial ideal (step (vii)). If the product of these ideals equals the intersection, this certifies that the basis of the annihilator generates the real radical. Correctness of the algorithm follows from Theorem 4.2.1.

Publications

The contributions of the thesis are based on the following works.

- [BM22b] **On the Effective Putinar's Positivstellensatz and Moment Approximation**, with Bernard Mourrain, accepted for publication in Mathematical Programming

- [BMP22] **On Łojasiewicz Inequalities and the Effective Putinar's Positivstellensatz**, with Bernard Mourrain and Adam Parusiński, in preparation

- [BM22a] **Exact Moment Representation in Polynomial Optimization**, with Bernard Mourrain, submitted for publication

- [BM21] **Computing Real Radicals by Moment Optimization**, with Bernard Mourrain, in: Proceedings of the 2021 on International Symposium on Symbolic and Algebraic Computation (ISSAC '21). Association for Computing Machinery, New York, NY, USA, 43-50. `https://doi.org/10.1145/3452143.3465541`

Structure of the thesis

This thesis is organized in the following chapters and sections.

- **Chapter 1** is dedicated to the description of the necessary background material, and to introduce the notation we will use through the article. Most of the content of the chapter is standard: when this is not the case, we provided precise references or presented directly the proofs.
  - **Section 1.1** is devoted the geometric and algebraic background. In particular, we introduce the basic notions of commutative algebra, complex algebraic geometry and real algebraic and semialgebraic geometry. Finally, we give an overview of representation theorems for positive polynomials.
  - **Section 1.2** is dedicated to duality theories of algebraic and geometric objects previously introduced. We describe the algebraic and topological dual cones of polynomials, and recall the basic results of convex duality.
  - **Section 1.3** gives an overview of the convex cone that we will use in the thesis, summarizing their properties. We introduce cones of polynomials positive on semialgebraic sets and their duals, recalling their basic properties. We also briefly introduce the cone of positive semidefinite matrices.
  - **Section 1.4** is devoted to the moment problem, with a particular focus on the truncated moment problem.
  - **Section 1.5** presents a general framework for duality in conic programming, and we describe how semidefinite programming can be expressed using it.
  - **Section 1.6** introduces the polynomial optimization problem. We introduce Lasserre's sum of squares and moment hierarchies, and then describe polynomial optimization problems and the Lasserre's hierarchies using the general framework for duality in conic programming.
- **Chapter 2** is devoted to the proof of an Effective version of Putinar's Positivstellensatz, and to its applications in polynomial optimization. This chapter is based on [BM22b] and [BMP22].
  - **Section 2.2** develops the proof of the Effective Putinar's Positivstellensatz. We present the principles of the proof in a general context, and then specialize to obtain the final bound.
  - **Section 2.3** describes Łojasiewicz inequalities in regular cases, giving different estimates for the Łojasiewicz constant and showing that the Łojasiewicz exponent is one.
  - **Section 2.4** gives an application of the previous results to prove the first general polynomial convergence of the Lasserre's hierarchies, with explicit constants and exponents. These bounds are improved for regular cases.
  - **Section 2.5** describes quantitatively the dual of the previous results, namely the convergence of normalized pseudo-moment sequences to moments of probability measures.

  - **Section 2.6** concludes the chapter, highlighting open questions and research perspectives.

- **Chapter 3** is dedicated to the study of different finite convergence properties in polynomial optimization, and in particular for the moment hierarchy. This chapter is based on [BM22a].
  - **Section 3.2** recalls the basic properties of Lasserre's hierarchies that we will need in the following.
  - **Section 3.3** introduces the main topics of the chapter, namely exactness for the sum of square and moment hierarchies, and flat truncation. Several examples are presented, that show how the properties of finite convergence, exactness and flat truncation are (and are not) related.
  - **Section 3.4** is dedicated to the study of geometric and algebraic properties of dual cones of truncated quadratic modules, and their projections. It is shown that the moment hierarchy coincides with an extended sum of squares hierarchy. The zero dimensional case is analyzed, and the connection between regularity of the points and the flat truncation highlighted.
  - **Section 3.5** applies the result of the previous section to polynomial optimization, and in particular to exactness of the moment hierarchy and flat truncation. It is shown the first sufficient and necessary condition for flat truncation property to hold, and show that this condition is generic. These results are applied to the cases of finite semialgebraic sets and polar ideals.
  - **Section 3.7** concludes the chapter, describing possible extensions and applications of the work.

- **Chapter 4** is devoted to the problem of solving real polynomial system in the positive dimensional case, i.e. computing real radicals. This chapter is based on [BM21].
  - **Section 4.2** introduces the main theoretical result connecting positive semidefinite Hankel operators and real radical computation, and summarize the necessary background in numerical algebraic geometry.
  - **Section 4.3** presents a new efficient algorithm to compute the basis of the annihilator of a positive linear functional, i.e. the kernel of the associated moment matrix.
  - **Section 4.4** is devoted to the reduction to the hypersurface case, and we present an algorithm based on optimization to verify if a complex irreducible component of the variety of the annihilator is totally real.
  - **Section 4.5** describes the final algorithm for the computation of equations for the irreducible components and the real radical.
  - **Section 4.6** concludes the chapter, with the presentation of examples and future perspectives.

Chapter 1

Preliminaries

In the thesis we investigate convergence and exact representation properties in Polynomial Optimization. The study of Polynomial Optimization is naturally at the intersection of different areas of mathematics: Optimization, Convex Geometry, Commutative Algebra and Algebraic Geometry, Semialgebraic Geometry and Functional Analysis.

In this chapter we recall the basic notions of these subjects, indicating all the references we used to develop the presentation, and we introduce the notations that we will use in the thesis. More precisely:

- **Section 1.1** is devoted the geometric and algebraic background. In Section 1.1.1 we recall basic definition from convex geometry. In Section 1.1.2 and Section 1.1.3 we introduce basic notions of commutative algebra, focusing on the polynomial ring with its grading and bases. In Section 1.1.4 we present the basics of complex algebraic geometry, while Section 1.1.5 deals with the real case. Finally, Section 1.1.6 offers an overview on semialgebraic geometry and on representation theorems for real polynomials that will be central in the thesis.

- **Section 1.2** is dedicated to duality theories of algebraic and geometric objects previously introduced. In Section 1.2.1 we consider algebraic dual spaces, in particular for the polynomial ring and its quotient rings. The presentation is completed by Section 1.1.3, where we describe the previous constructions choosing a basis. In Section 1.2.3 we describe the topological dual, and we recall the main results of convex duality in Section 1.2.4.

- **Section 1.3** gives an overview of the convex cone that we will use in the thesis, summarizing their properties. We start describing positive polynomials (Section 1.3.1) and their dual counterpart, Borel measures (Section 1.3.2). We introduce positive semidefinite matrices (Section 1.3.3), and highlight their relationships with sums of squares polynomials (Section 1.3.4) and quadratic modules (Section 1.3.5). Section 1.3.6 is devoted to dual cones of quadratic modules. Finally, in Section 1.3.7 and Section 1.3.8 we present the truncated versions of the objects previously introduced.

- **Section 1.4** is devoted to the moment problem, with a particular focus on the truncated moment problem (Section 1.4.1).

- **Section 1.5** presents a general framework for duality in conic programming, and we describe how semidefinite programming can be expressed using it (Section 1.5.1).

- **Section 1.6** introduces the polynomial optimization problem. We introduce Lasserre's sum of squares hierarchy in Section 1.6.1, and Lasserre's moment hierarchy in Section 1.6.2. We then describe polynomial optimization (Section 1.6.3) and the Lasserre's hierarchies (Section 1.6.4) using the general framework for duality in conic programming.

1.1 Geometry and algebra

1.1.1 Convex geometry

Let Z be a vector space over $\mathbb{R}$. The main examples that we will consider in the paper are the $\mathbb{R}$-algebra of polynomials, polynomials of bounded degree and their duals. Hereafter we briefly summarize basic definitions and results of convex geometry and refer to [Bar02; Roc97; Rud91] for proofs and more details. See also Section 1.2.3 and Section 1.2.4 for convexity properties arising from considering dual spaces.

Definition 1.1.1. We say that $C \subset Z$ is *convex* if for all $s \in [0,1]$ and $v, w \in C$, we have $sv + (1-s)w \in C$. Given $v_1, \ldots, v_r \in V$, a linear combination $s_1 v_1 + \cdots + s_r v_r$ is a *convex combination* of $v_1, \ldots, v_r$ if $s_i \geq 0$ and $\sum_i s_i = 1$.

Given a subset $B \subset Z$, we define the *convex hull* of B as the set of convex combinations of elements of B:

$$\operatorname{conv}(B) := \Big\{ s_1 v_1 + \cdots + s_r v_r \in Z \mid r \in \mathbb{N},\ v_i \in B,\ s_i \geq 0,\ \sum_{i=1}^{r} s_i = 1 \Big\}.$$

Notice that $\operatorname{conv}(B)$ is the smallest convex subset of Z containing B. In the definition of convex hull, there is no upper bound on the number r of addenda in the convex combination. In the finite dimensional case, a general upper bound is given by the Caratheodory's theorem.

Theorem 1.1.2 (Caratheodory's Theorem)**.** *If Z is an n-dimensional vector space, then:*

$$\operatorname{conv}(B) = \Big\{ s_1 v_1 + \cdots + s_{n+1} v_{n+1} \in Z \mid v_i \in B,\ s_i \geq 0,\ \sum_{i=1}^{n+1} s_i = 1 \Big\}.$$

A subset $F \subset C$ of a closed convex set C is called *face* if, for $s \in [0,1]$ and $v, w \in C$, $sv + (1-s)w \in F$ implies $v, w \in F$. We say that $u \in C$ is an *extremal point* if, for all $s \in (0,1)$ and $v, w \in C$, $sv + (1-s)w = u$ implies $u = v = w$, or equivalently if $\{u\}$ is a face of C. A face F of C is called *exposed* if there exists an affine hyperplane H such that $F = C \cap H$.

Among all convex sets, in particular we are interested in *convex cones*.

Definition 1.1.3. We call $C \subset Z$ a *cone* if, for all $s \in \mathbb{R}_{\geq 0}$ and $v \in C$, we have $sv \in C$. We call $C \subset Z$ a *convex cone* if for all $s, t \in \mathbb{R}_{\geq 0}$ and $v, w \in C$, we have $sv + tw \in C$.

Notice that C is a convex cone if and only if it is convex and it is a cone. Given $v_1, \ldots, v_r \in V$, a linear combination $s_1 v_1 + \cdots + s_r v_r$ is a *conic combination* of $v_1, \ldots, v_r$ if $s_1 \geq 0$. Given a subset $B \subset Z$, we define the *conic hull* of B as the set of conic combinations of elements of B:

$$\operatorname{cone}(B) := \Big\{ s_1 v_1 + \cdots + s_r v_r \in Z \mid r \in \mathbb{N},\ v_i \in B,\ s_1 \geq 0 \Big\}.$$

Notice that cone(B) is the smallest convex cone in Z containing B. A cone C is called *pointed* if $0 \in C$. Through the thesis, all the cones that we will consider will be pointed. We say that a point $v \in C$ spans an *extremal ray* of the convex cone C if the ray $\mathbb{R}_{\geq 0}v$ spanned by v is a face of C.

The smallest linear space containing a convex cone C is $C - C = \{v - w \in Z \mid v, w \in c\}$. The biggest linear space contained in C is $C \cap -C$, and it is sometimes called the *lineality space* of C. The lineality space will play an important role especially when $C = Q$ is a quadratic module in the real polynomial ring, see Section 1.1.6. In this case $Q \cap -Q$ is called the *support* of Q.

1.1.2 Commutative algebra and polynomials

We recall the basic definition of commutative algebra to fix the notations, and refer to [AM94; Eis04] for more details. All the rings that we will consider will be commutative and with identity.

Let R be a commutative ring with identity. A subset $I \subset R$ is an *ideal* if $I + I \subset I$ (that is, I closed under addition) and $R \cdot I \subset I$ (that is, I is closed under multiplication by R). For an ideal I, we can consider the quotient ring $\frac{R}{I}$.

We say that an ideal $\mathfrak{p}$ is *prime* if $\mathfrak{p} \neq R$ and for $f, g \in R$, $fg \in \mathfrak{p}$ implies $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. Equivalently, $\mathfrak{p}$ is prime if $\frac{R}{\mathfrak{p}}$ is a nonzero integral domain (a ring is called integral domain if the product of two nonzero elements is always nonzero).

We say that an ideal I is *radical* if $\frac{R}{I}$ is a reduced ring: if $x \in \frac{R}{I}$ and $x^n = 0$ for some $n \in N$, then $x = 0$.

The radical $\sqrt{I}$ of an ideal I is the smallest radical ideal containing I. It is equal to the intersection of all (minimal) prime ideals $\mathfrak{p}$ containing I, or explicitely:

$$\sqrt{I} = \{x \in R \mid \exists n \in \mathbb{N} \text{ such that } x^n \in I\}.$$

If D is an integral domain, we write Quot(D) for the quotient or fraction field.

Let $\mathbb{K}$ be a field, V a finite dimensional vector space of dimension n over $\mathbb{K}$ and let $S^\bullet(V)$ the symmetric algebra (the quotient of the tensor algebra $T^\bullet(V) = \bigoplus_{d\in\mathbb{N}} V^{\otimes d}$ modulo the commutation relations $v \otimes w - w \otimes v$). A basis $\mathbf{x} = \{x_1, \ldots, x_n\}$ of V gives naturally isomorphisms $V \cong \mathbb{K}^n$ and $S^\bullet(V) \cong \mathbb{K}[x_1, \ldots, x_n] =: \mathbb{K}[\mathbf{x}]$, the usual multivariate polynomial algebra. $S^\bullet(V)$ is a graded algebra: $S^\bullet(V) = \bigoplus_{d\in\mathbb{N}} S^d(V)$, where $S^d(V)$ is the d-th symmetric power of V. Concretely, $\mathbb{K}[\mathbf{x}] = \bigoplus_{d\in\mathbb{N}} \mathcal{H}_{n,d} = \bigcup_{d\in\mathbb{N}} \mathbb{K}[\mathbf{x}]_d$, where $\mathcal{H}_{n,d} \cong S^d(V)$ are n-variate d-homogeneous polynomials (or *forms*), and $\mathbb{K}[\mathbf{x}]_d := \bigoplus_{i\in\{0,\ldots,d\}} \mathcal{H}_{n,i}$ are the polynomials of total degree $\leq d$. For $f \in \mathbb{K}[\mathbf{x}]$, we denote $\deg f$ the smallest integer d such that $f \in \mathbb{K}[\mathbf{x}]_d$, and we call it *degree* of f. See also Section 1.1.3.

For sake of simplicity we will write $\mathbb{K}[\mathbf{x}]$ rather than $S^\bullet(V)$, implicitly assuming the use of the monomial basis (see Section 1.1.3) and using the grading described above. But all our constructions would work for any other choice of basis.

1.1.3 Monomials, grading and bases

In this section we focus on the $\mathbb{K}$-algebra $\mathbb{K}[\mathbf{x}]$, and describe it choosing the basis of monomials and the grading of the total degree.

A basis $\mathbf{x} = (x_1, \dots, x_n)$ of the $\mathbb{K}$-vector space V gives an isomorphism $S^\bullet(V) \cong \mathbb{K}[\mathbf{x}]$. If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is a multi-index, we denote $\mathbf{x}^\alpha := \prod_{i=1}^n x_i^{\alpha_i}$ and call $\mathbf{x}^\alpha$ a *monomial*. The sequence of all monomials is a basis of $\mathbb{K}[\mathbf{x}]$ over $\mathbb{K}$: any $f \in \mathbb{K}[\mathbf{x}]$ can be written in a unique way as $f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha \mathbf{x}^\alpha$, where $f_\alpha \in \mathbb{K}$ are equal to zero for all but finitely many indexes. The *degree* of f is then equal to $\deg f = \max\{|\alpha| : f_\alpha \neq 0\}$.

Since the polynomial ring is Noetherian, any ideal $I \subset \mathbb{K}[\mathbf{x}]$ is *finitely generated*: for any ideal I, there exist $f_1, \dots, f_r \in \mathbb{K}[\mathbf{x}]$ such that I is equal to the ideal generated by $f_1, \dots f_r$:

$$I = (f_1, \dots, f_r) := \{\sum_{i=1}^r h_i f_i \mid h_i \in \mathbb{K}[\mathbf{x}]\},$$

and we call $f_1, \dots f_r$ a *basis* of I.

We are in particular interested in the relationship between bases and the grading of $\mathbb{K}[\mathbf{x}]$.

Definition 1.1.4. We say that the tuple of polynomials $\mathbf{h} = h_1, \dots, h_s$ is a *graded basis* of an ideal I if for all $p \in I$, there exists $q_i \in \mathbb{K}[\mathbf{x}]$ with $\deg(q_i) \leq \deg(p) - \deg(h_i)$ such that $p = \sum_{i=1}^s h_i q_i$.

Equivalently, if we define, for $t \in \mathbb{N}$:

$$\langle \mathbf{h} \rangle_t := \{p = \sum_{i=1}^s h_i q_i \mid \deg(q_i) \leq t - \deg(h_i)\}$$

and

$$I_t := \{f \in I \mid \deg f \leq t\} = I \cap \mathbb{K}[\mathbf{x}]_t,$$

then $\mathbf{h}$ is graded basis if $\langle \mathbf{h} \rangle_t = I_t$ for all t. We call $\langle \mathbf{h} \rangle_{\leq t}$ the *truncated ideal* in degree t generated by $\mathbf{h}$. If $\mathbf{h}$ is not a graded basis, the inclusion $\langle \mathbf{h} \rangle_t \subset I_t$ is strict.

We briefly show how graded bases relate to other kind of bases: in particular to Groebner bases (see e.g. [CLO15]) and to border bases (see e.g. [MT05]).

Definition 1.1.5. An *monomial order* on the set of monomials is a total order $\prec$ that is compatible with multiplication: if $\mathbf{x}^\alpha \prec \mathbf{x}^\beta$ then $\mathbf{x}^\alpha \mathbf{x}^\gamma \prec \mathbf{x}^\beta \mathbf{x}^\gamma$ for all $\alpha, \beta, \gamma \in \mathbb{N}^n$. The *leading term* of a polynomial $f = \sum_\alpha f_\alpha \mathbf{x}^\alpha$ is the biggest monomial $\mathbf{x}^\alpha$ with respect to $\prec$ such that $f_\alpha \neq 0$. A *Groebner basis* of an ideal I is a finite tuple of polynomials $\mathbf{h}$ in I such that the ideal generated from the leading terms of I is equal to the ideal generated from the leading terms of $\mathbf{h}$.

We say that a monomial ordering *refines* the total degree if, for $\alpha, \beta \in \mathbb{N}^n$, $|\alpha| < |\beta|$ implies $\mathbf{x}^\alpha \prec \mathbf{x}^\beta$. A graded basis of an ideal $I = (\mathbf{h})$ can be computed as a Groebner basis using a monomial ordering $\prec$, which refines the degree ordering.

1.1.4 Varieties

In this thesis, we are mainly interested in the affine varieties defined over the complex and real numbers. Hereafter we introduce the concepts we need for complex affine varieties, and refer to [Sha13a; Sha13b] for more details (there, our varieties are called *closed subsets*). We will describe in Section 1.1.5 the real case.

We introduce affine complex varieties in the naive way, as subsets of $\mathbb{C}^n$. We say that $X \subset \mathbb{C}^n$ is a *complex affine variety*, or simply a *variety*, when X is the common zero locus of a family of polynomials $f_j\colon j \in J$ with complex coefficients. In this case we write $X = \mathcal{V}_{\mathbb{C}}(f_j\colon j \in J) = \{x \in \mathbb{C}^n \mid f_j(x) = 0\ \forall j \in J\}$. It is easy to see that, if $I = (f_j\colon j \in J)$ is the ideal generated by $f_j\colon j \in J$, we have $\mathcal{V}_{\mathbb{C}}(f_j\colon j \in J) = \mathcal{V}_{\mathbb{C}}(I)$. Affine varieties are the closed sets of a topology, called the *Zariski topology*. This topology is defined in an equivalent way when we replace $\mathbb{C}$ by any other field $\mathbb{K}$. Conversely, given a subset $B \subset \mathbb{C}^n$, one can consider the *vanishing ideal* of B: $\mathcal{I}_{\mathbb{C}}(B) = \{f \in \mathbb{C}[\mathbf{x}] \mid f(x) = 0\ \forall x \in B\}$. If we denote $\mathrm{cl}(B)$ the Zariski closure of B (that is, the smallest set containing B that is the common zero locus of polynomial equations), then it is easy to show $\mathcal{I}_{\mathbb{C}}(B) = \mathcal{I}_{\mathbb{C}}(\mathrm{cl}(B))$. Notice that $\mathcal{I}_{\mathbb{C}}(B)$ is by definition a radical ideal (but it can be not prime).

The above correspondence is made precise by the Hilbert's Nullstellensatz.

Theorem 1.1.6 (Hilbert's Nullstellensatz [Eis04; AM94]). *If $I \subset \mathbb{C}[\mathbf{x}]$ is an ideal, then $\mathcal{I}_{\mathbb{C}}(\mathcal{V}_{\mathbb{C}}(I)) = \sqrt{I}$. Moreover, the correspondence $X \mapsto \mathcal{I}_{\mathbb{C}}(X)$ and $I \mapsto \mathcal{V}_{\mathbb{C}}(I)$ induces an inclusion reversing bijection between radical ideals in $\mathbb{C}[\mathbf{x}]$ and affine complex varieties in $\mathbb{C}^n$.*

Any variety X is naturally equipped with its *coordinate ring* $\mathbb{C}[X]$, i.e. the ring of polynomial functions on X. If $I = \mathcal{I}_{\mathbb{C}}(X)$, there is an isomorphism $\mathbb{C}[X] \cong \frac{\mathbb{C}[\mathbf{x}]}{I}$.

Using the language of schemes, complex affine varieties can be described as the spectrum of a reduced finitely generated $\mathbb{C}$-algebra A, and $A = \mathbb{C}[X]$ is the coordinate ring. The spectrum of A, whose points are the prime ideals in A, is equipped with the Zariski topology. The closed points are the maximal ideals $\mathfrak{m}$ of A, that are in correspondence with the homomorphisms $\phi\colon A \to \mathbb{C}$ (called $\mathbb{C}$-rational points) via $\mathfrak{m} = \ker\phi$. Concretely, fixing a representation $A \cong \frac{\mathbb{C}[\mathbf{x}]}{\mathcal{I}_{\mathbb{C}}(X)}$ the $\mathbb{C}$-rational points are the set of *evaluations* $\mathbf{e}_x$ at points $x \in X$ (see also Section 1.2.1) and there is a one to one correspondence between $\mathbb{C}$-rational points (or evaluations), maximal ideals of A and points $x \in X$.

Since in the following no other advantage arise using the language of schemes, we will keep using the classical, naive definition of varieties.

We now introduce the topological property of *irreducibility*. We say that a topological space X is *irreducible* if it cannot be written as the union of two proper subsets: $X = X_1 \cup X_2$, with X_1, X_2 closed implies $X = X_1$ or $X = X_2$. Equivalently, X is irreducible if all the nonempty subsets of X are dense. An *irreducible component* of X is a maximal closed subspace of X that is irreducible in X (with the subspace topology).

A variety X (equipped with the Zariski topology) is irreducible if and only if $\mathcal{I}(X) = \mathfrak{p}$ is a prime ideal. Any variety X can be written as the union of irreducible components $X = X_1 \cup \cdots \cup X_r$, essentially in a unique way, that correspond to the decomposition $\mathcal{I}(X) = \bigcap_{i=1}^r \mathfrak{p}_i$ of the radical ideal $\mathcal{I}(X)$ as intersection of minimal prime ideals lying over $\mathcal{I}(X)$.

For an irreducible variety X, the vanishing ideal $\mathcal{I}(X) = \mathfrak{p}$ is prime and the coordinate ring $\mathbb{C}[X] \cong \frac{\mathbb{C}[\mathbf{x}]}{\mathfrak{p}}$ is a domain. The field of fractions of $\mathbb{C}[X]$ is called the *function field* of *field of rational functions* on X and it is denoted $\mathbb{C}(X)$. An element $\phi \in \mathbb{C}(X)$ defines a *rational function* $x \mapsto \phi(x)$ defined on the (open, dense) *domain* U of ϕ where ϕ is regular. Explicitly, $U = \bigcup_i X \setminus \mathcal{V}(g_i)$, where $f_i/g_i = \phi$ is any representative of ϕ.

A *rational map* is a map $\phi = (\phi_1, \ldots, \phi_m)$ defined by an m-tuple of rational functions $\phi_i \in \mathbb{C}(X)$. Notice that ϕ is not defined everywhere, but only on a dense open subset of X. We write $\phi\colon X \dashrightarrow \mathbb{C}^m$ and, if the image of ϕ is included in the variety $Y \subset \mathbb{A}^m_{\mathbb{C}}$, we write $\phi\colon X \dashrightarrow Y$. We say that a rational map $\phi\colon X \dashrightarrow Y$ is *birational* if $\phi(X)$ is dense in Y, and there exists a rational map $\psi\colon Y \dashrightarrow X$ such that $\psi(Y)$ is dense in X and the maps $\phi\circ\psi$ and $\psi\circ\phi$ are identities (on the respective domains of definition). Equivalently, $\phi\colon X \dashrightarrow Y$ is birational if and only if $\phi^*\colon \mathbb{C}(Y)\to\mathbb{C}(X)$, $u\mapsto u\circ\phi$ is an isomorphism. We say that two varieties X and Y are *birational* if there exists a birational map $\phi\colon Y \dashrightarrow X$, or equivalently if the two function fields $\mathbb{C}(X)$ and $\mathbb{C}(Y)$ are isomorphic.

We now describe the main birational invariant of algebraic varieties, the dimension, and then introduce singular points.

Proposition 1.1.7. *Let X be an irreducible complex algebraic variety. Then the following are equal:*

- *The trascendence degree d of the quotient field* $\mathbb{C}(X) = \mathrm{Quot}(\mathbb{C}[X])$ *over C;*
- *The Krull dimension of* $\mathbb{C}[X]\cong\mathbb{C}[\mathbf{x}]/\mathcal{I}_{\mathbb{C}}(X)$, *i.e. the maximal length of chain of prime ideals ideals in* $\mathbb{C}[X]$*:*
$$0=\mathfrak{p}_0\subsetneq\mathfrak{p}_1\subsetneq\cdots\subsetneq\mathfrak{p}_d$$
- *The maximal length d of a chain of irreducible subvarieties of X:*
$$X_0\subsetneq X_1\subsetneq\cdots\subsetneq X_d=X$$

Proof. See [Sha13a; AM94] or [Mar08, app. 2]. □

Definition 1.1.8. Let X be an irreducible complex algebraic variety. Then we call the *dimension* of X the natural number d in Proposition 1.1.7, and we denote it $d=\dim X$. If X is a complex algebraic variety (not necessarily irreducible), where $X=\bigcup_i X_i$ is a decomposition in irreducible components, then we define the *dimension* of X as the maximal dimension of its irreducible components: $\dim X=\max_i\dim X_i$.

Definition 1.1.9. Let X be a complex algebraic variety. Let $f_1,\dots f_m$ be generators of the vanishing ideal of X, i.e. $(f_1,\dots f_m)=\mathcal{I}_{\mathbb{C}}(X)\subset\mathbb{C}[\mathbf{x}]$. Then we say that $\xi\in X\subset\mathbb{C}^n$ is a *smooth* or *nonsingular* point if $\operatorname{rank}\operatorname{Jac}(f_1,\dots,f_m)(\xi)=n-\dim X$. Otherwise, we say that ξ is a *singular* point.

See [Sha13a] for a more intrinsic definition of smooth and singular points.

1.1.5 Real algebra and geometry

In this section we introduce the notations of real algebra and real algebraic geometry that we will need through the thesis, and refer to [BCR98; Mar08; Man20] for more details.

As in the complex case, we introduce real affine varieties in the naive way, and refer to [Man20, sec. 2.4] for a detailed discussion of different definitions that can be found in the literature. As in the complex case, our real varieties will be reduced but not necessarily irreducible.

Let $\mathbb{R}^n$ denote the affine n-dimensional space over $\mathbb{R}$. We say that $X\subset\mathbb{R}^n$ is a *real affine variety* when X is the common zero locus of a family of polynomials $f_j\colon j\in J$ with real coefficients. In this case we write $X=\mathcal{V}_{\mathbb{R}}(f_j\colon j\in J)=\{x\in\mathbb{R}^n\mid f_j(x)=0\ \forall j\in J\}$. As is the complex case, if I is the (real) ideal generated by $f_j\colon j\in J$, we have $\mathcal{V}_{\mathbb{R}}(f_j\colon j\in J)=\mathcal{V}_{\mathbb{R}}(I)$. The real affine varieties are the closed sets of a topology, called the *Zariski topology*. Using this topology, one can define *irreducible* real varieties as in the complex case, and we have an essentially unique decomposition of every real variety X as union of its irreducible components: $X = \bigcup_i X_i$.

But of course $\mathbb{R}^n = \mathbb{R}^n$ can also be equipped with the Euclidean topology, and the same is true for real varieties $X \subset \mathbb{R}^n$ with the subspace topology. Therefore, for $B \subset \mathbb{R}^n$, we denote:

- $\overline{B} \subset \mathbb{R}^n$ for the closure of B in the Euclidean topology;
- $\mathrm{cl}(B) \subset \mathbb{C}^n$ for the closure of B in the complex Zariski topology, or in other words for the smallest set of complex points that contains B and can be described as the common zero locus of complex polynomials.

For a real variety $X \subset \mathbb{R}^n$, we can consider the *complexification* $X_{\mathbb{C}} = \mathrm{cl}(X)$ of X. $X_{\mathbb{C}}$ is the smallest complex variety containing X. On the other hand, given a complex variety $X \subset \mathbb{C}^n$, we denote $X_{\mathbb{R}} = X \cap \mathbb{R}^n$ the set of real points.

As in the complex case, we define the *dimension* $\dim X$ of a real variety X as The maximal length d of a chain of (real) irreducible subvarieties of X:

$$X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_d = X.$$

For a more detailed discussion about the real dimension, especially in connection with the Euclidean topology, see [BCR98].

The relation between complexification, irreducible components and dimension have been studied by Whitney: in particular, the dimension of a real variety X (with the real Zariski topology) is equal to the dimension of the complexification $X_{\mathbb{C}}$ (with the complex Zariski topology), and there is a correspondence between their irreducible decompositions.

Theorem 1.1.10 (Whitney, [Whi57]). *Let X be a real algebraic variety. Then the dimension of X (as a real variety) is equal to the dimension of $X_{\mathbb{C}}$ (as a complex variety):* $\dim X = \dim X_{\mathbb{C}}$. *Moreover, if $X = \bigcup_i X_i$ is the decomposition in irreducible components of X, then $X_{\mathbb{C}} = \bigcup_i (X_i)_{\mathbb{C}}$ is the decomposition in irreducible components of $X_{\mathbb{C}}$.*

Given a subset $B \subset \mathbb{R}^n$, one can consider the *real vanishing ideal* of B: $\mathcal{I}_{\mathbb{R}}(B) = \{f \in \mathbb{R}[\mathbf{x}] \mid f(x) = 0 \ \forall x \in B\}$. We introduce now the class of ideals in $\mathbb{R}[\mathbf{x}]$, that play the same role of radical ideals in the complex case.

Definition 1.1.11. We say that an ideal $I \subset \mathbb{R}[x_1,\dots,x_n]$ is *real* if, for any $f_i,\dots,f_p \in \mathbb{R}[x_1,\dots,x_n]$, $f_1^2 + \cdots + f_p^2 \in I \Rightarrow f_i \in I$ for all i.

We see in the definition of real ideals the important role of *Sums of Squares* polynomials, that we denote:

$$\Sigma^2 = \Sigma\mathbb{R}[\mathbf{x}]^2 := \left\{ f \in \mathbb{R}[\mathbf{X}] \mid \exists r \in \mathbb{N},\ g_i \in \mathbb{R}[\mathbf{X}] \colon f = g_1^2 + \cdots + g_r^2 \right\}.$$

Now we can introduce real radical ideals.

Definition 1.1.12 (Real Radical). Let $I \subset \mathbb{R}[\mathbf{X}]$ be an ideal. The *real radical* of I is the ideal:

$$\sqrt[\mathbb{R}]{I} := \{ f \in \mathbb{R}[\mathbf{X}] \mid \exists m \in \mathbb{N},\ s \in \Sigma^2 \text{ with } f^{2m} + s \in I \}.$$

We give an equivalent description of the real radical that will be useful in the thesis.

Lemma 1.1.13. *Let $I \subset \mathbb{R}[\mathbf{x}]$ be an ideal. Then:*

$$\sqrt[\mathbb{R}]{I} = \sqrt{(I + \Sigma^2) \cap -(I + \Sigma^2)}.$$

Proof. $(I + \Sigma^2) \cap -(I + \Sigma^2)$ is an ideal (see Lemma 1.1.24), and the equality $\{f \in \mathbb{R}[\mathbf{X}] \mid \exists m \in \mathbb{N},\ s \in \Sigma^2 \text{ with } f^{2m} + s \in I\} = \sqrt{(I + \Sigma^2) \cap -(I + \Sigma^2)}$ is easy to prove:

$$\begin{aligned} f^{2m} + s \in I &\iff -f^{2m} \in \Sigma^2 + I \\ &\iff f^{2m} \in (I + \Sigma^2) \cap -(I + \Sigma^2) \\ &\iff f \in \sqrt{(I + \Sigma^2) \cap -(I + \Sigma^2)}. \end{aligned}$$

□

The real radical of ideal $I \subset \mathbb{R}[\mathbf{x}]$ is the smallest real ideal containing I. It is also the intersection of the minimial prime ideals $\mathfrak{p}$ lying over I that are real: $\sqrt[\mathbb{R}]{I} = \bigcap_{\mathfrak{p} \supset I,\ \mathfrak{p} \text{ real prime}} \mathfrak{p}$.

The analogy of real ideals with radical ideals in the complex case is made precise by the Real Nullstellensatz.

Theorem 1.1.14 (Real Nullstellensatz, [BCR98, th. 4.1.4, cor. 4.1.8])**.** *If $I \subset \mathbb{R}[\mathbf{x}]$ is an ideal, then $\mathcal{I}_{\mathbb{R}}(\mathcal{V}_{\mathbb{R}}(I)) = \sqrt[\mathbb{R}]{I}$. Moreover, the correspondence $X \mapsto \mathcal{I}_{\mathbb{R}}(X)$ and $I \mapsto \mathcal{V}_{\mathbb{R}}(I)$ induces an inclusion reversing bijection between real ideals in $\mathbb{R}[\mathbf{x}]$ and affine real varieties in $\mathbb{R}^n$.*

1.1.6 Semialgebraic geometry and positivity

Up to now we did not exploit one of the main properties of real numbers: the ordering $\geq$. Thanks to the ordering, we can consider a larger class of subsets of $\mathbb{R}^n$ than the one defined by polynomial equations: those that are defined by finitely many polynomial *inequalites*. These subsets are called *semialgebraic sets*. We refer to [BCR98; PD01; Mar08; Pow21] for more details about semialgebraic geometry and positivity. We are in particular interested in the case where those inequalities are not strict.

Definition 1.1.15. Let $\mathbf{g} = g_1, \ldots, g_r$ be a finite tuple of real polynomials, $g_i \in \mathbb{R}[\mathbf{x}]$. The *basic closed semialgebraic set* defined by $\mathbf{g}$ is:

$$\mathcal{S}(\mathbf{g}) = \mathcal{S}(g_1, \ldots, g_r) = \left\{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots g_r(x) \geq 0\right\}$$

The *dimension* of a (basic, closed) semialgebraic set S is the dimension of the real Zariski closure of S in $\mathbb{R}^n$. See [BCR98] for a more details on the dimension of semialgebraic sets.

We follow the French tradition and we say that a polynomial f is:

- *positive* or *non-negative* on a domain D if $f(x) \geq 0$ for all $x \in D$;
- *strictly positive* on a domain D if $f(x) > 0$ for all $x \in D$.

We write also $f \geq 0$ on D and $f > 0$ on D respectively. We denote

$$\mathrm{Pos}(D) := \{f \in \mathbb{R}[\mathbf{x}] \mid f(x) \geq 0 \text{ for all } x \in D\}$$

the positive polynomials on a domain D. It is easy to show that $\mathrm{Pos}(D)$ is a convex cone in $\mathbb{R}[\mathbf{x}]$.

A special type on globally positive polynomials are the *Sums of Squares polynomials* (SoS):

$$\Sigma^2 = \Sigma\mathbb{R}[\mathbf{x}]^2 = \left\{ f \in \mathbb{R}[\mathbf{X}] \mid \exists r \in \mathbb{N},\ g_i \in \mathbb{R}[\mathbf{X}] \colon f = g_1^2 + \cdots + g_r^2 \right\}.$$

Σ^2 is a convex cone in $\mathbb{R}[\mathbf{x}]$, and by definition $\Sigma^2 \subset \mathrm{Pos}(\mathbb{R}^n)$. Furthermore, notice that the smallest linear space containing Σ^2, namely $\Sigma^2 - \Sigma^2$, is equal to $\mathbb{R}[\mathbf{x}]$. Indeed, any $f \in \mathbb{R}[\mathbf{x}]$ can be written as

$$f = \frac{1}{4}\left((f+1)^2 - (f-1)^2\right) \in \Sigma^2 - \Sigma^2.$$

Moreover, the cone Σ^2 is pointed: $\Sigma^2 \cap -\Sigma^2 = \{0\}$, since a polynomial $f \in \Sigma^2 \cap -\Sigma^2$ is identically zero on $\mathbb{R}^n$, and thus $f = 0$. Finally, Σ^2 is a *closed* convex cone (as a subset of $\mathbb{R}[\mathbf{x}]$ with the finest locally convex topology): indeed if $f \in \Sigma^2 \cap \mathbb{R}[\mathbf{x}]_{2d}$, then f is a sum of squares of polynomials of degree $\leq d$ (since the terms of highest degree cannot cancel). Then one can proceed as in [Lau09, sec. 3.8].

Let $S = \mathcal{S}(\mathbf{g})$ be a basic closed semialgebraic set. We can use SoS polynomials to find other subcones of $\mathrm{Pos}(S)$: *quadratic modules* and *preorderings*.

Definition 1.1.16. $Q \subset \mathbb{R}[\mathbf{X}]$ is called *quadratic module* if $1 \in Q$, $\Sigma^2 \cdot Q \subset Q$ and $Q + Q \subset Q$. We say that a quadratic module Q is *finitely generated* if $\exists g_1 \dots g_r \in \mathbb{R}[\mathbf{x}]$ such that:

$$Q = \mathcal{Q}(\mathbf{g}) = \mathcal{Q}(g_1,\dots,g_r) := \Sigma^2 + \Sigma^2 \cdot g_1 + \cdots + \Sigma^2 \cdot g_r$$

(it is the smallest quadratic module containing $g_1,\dots,g_r$).

Definition 1.1.17. $O \subset \mathbb{R}[\mathbf{X}]$ is called *preordering* if $1 \in O$, $\Sigma^2 \cdot O \subset O$, $O + O \subset O$ and $O \cdot O \subset O$. We say that a preordering O is *finitely generated* if $\exists g_1 \dots g_r \in \mathbb{R}[\mathbf{x}]$ such that:

$$O = \mathcal{O}(\mathbf{g}) = \mathcal{O}(g_1,\dots,g_r) := \Sigma^2 + \Sigma^2 \cdot g_1 + \cdots + \Sigma^2 \cdot g_r + \Sigma^2 \cdot g_1 g_2 + \cdots + \Sigma^2 \cdot g_1 \cdots g_r$$

(it is the smallest preordering containing $g_1,\dots,g_r$).

In particular, any preordering is a quadratic module, and notice that, by definition, $\mathcal{Q}(\mathbf{g}) \subset \mathcal{O}(\mathbf{g}) \subset \mathrm{Pos}(\mathcal{S}(\mathbf{g}))$.

Hereafter we introduce convenient notations to relate quadratic modules, preorderings and ideals. Given a finite tuple of polynomials $\mathbf{g} = g_1,\dots,g_r$, we define $\Pi\mathbf{g} := \prod_{j\in J} g_j \colon J \subset \{1,\dots,r\} = g_1,\dots,g_r, g_1g_2,\dots,g_1\cdots g_r$, the tuple of all the products of the g_i's, and $\pm\mathbf{g} := g_1,-g_1,\dots,g_r,-g_r$ (we are interested in this definition because any equation $g = 0$ can of course be realized as $g \geq 0$ and $-g \geq 0$). Then, for tuples of polynomials $\mathbf{g}$ and $\mathbf{h}$:

- $\mathcal{O}(\mathbf{g}) = \mathcal{Q}(\Pi\mathbf{g})$ (the preordering generated by $\mathbf{g}$ is the quadratic module generated by the tuple $\Pi\mathbf{g}$);
- $\mathcal{Q}(\pm\mathbf{h}) = \Sigma^2 + (\Sigma^2 - \Sigma^2)h_1 + \cdots + (\Sigma^2 - \Sigma^2)h_1 = \Sigma^2 + (\mathbf{h})$ (the quadratic module generated by equations is the ideal of the equations plus SoS polynomials);
- $\mathcal{Q}(\mathbf{g}, \pm\mathbf{h}) = \mathcal{Q}(\mathbf{g}) + (\mathbf{h})$.

It is a natural question to describe explicitly the cone $\mathrm{Pos}(S)$ of positive polynomials on a basic closed semialgebraic set S. A complete description of $\mathrm{Pos}(S)$ can be obtained using preorderings, and it is given by the Krivine-Stengle Positivstellensatz:

Theorem 1.1.18 ([Kri64],[Ste74]). *Let* $\mathbf{g}$ *be a tuple of polynomials and* $S = S(\mathbf{g})$*. Then:*

$$\mathrm{Pos}(S) = \{p \in \mathbb{R}[\mathbf{X}] \mid \exists s \in \mathbb{N},\ q_1, q_2 \in \mathcal{O}(\mathbf{g}) \text{ such that } q_1 p = p^{2s} + q_2\}$$

This result is an extension of Artin's theorem [Art27], stating that globally positive polynomials are ratio of two SoS polynomials. But it induces a *denominator* in the representation of a positive polynomial.

For a general tuple $\mathbf{g}$ and $n > 1$, positive polynomials on $S = S(\mathbf{g})$ do not all belong to the quadratic module $Q(\mathbf{g})$ or even to the preordering $\mathcal{O}(\mathbf{g})$, and it is natural to ask whether the convex cone $Q(\mathbf{g})$ (resp. $\mathcal{O}(\mathbf{g})$) is a good inner-approximation of $\mathrm{Pos}(S)$. We are interested mainly in the case when S is compact, and in particular when $\mathcal{Q}(\mathbf{g})$ is *Archimedean*.

Lemma 1.1.19. *Let* $Q \subset \mathbb{R}[\mathbf{x}]$ *be a quadratic module, and denote* $\|\mathbf{x}\|_2^2 := x_1^2 + \cdots + x_n^2 \in \mathbb{R}[\mathbf{x}]$*. Then the following are equivalent:*

(i) for all $f \in \mathbb{R}[\mathbf{x}]$*, there exists* $n \in \mathbb{N}_{>0}$ *such that* $f + n \in Q$*;*

(ii) there exists $n \in \mathbb{N}_{>0}$ *such that* $n^2 - \|\mathbf{x}\|_2^2 \in Q$*;*

(iii) there exists $n \in \mathbb{N}_{>0}$ *such that* $n \pm x_i \in Q$*.*

Proof. See [Mar08, cor. 5.2.4]. □

Definition 1.1.20. We say that a quadratic module Q is *Archimedean* if any of the equivalent conditions in Lemma 1.1.19 is satisfied.

Notice in particular that, if $\mathcal{Q}(\mathbf{g})$ is Archimedean, then $\mathcal{S}(\mathcal{Q}(\mathbf{g}))$ is compact. The converse in not true for quadratic modules, see [PD01, ex. 6.3.1]. But for preorderings, it is true, as a consequence of the celebrated Schmüdgen's Positivstellensatz, describing strictly positive polynomials on a compact semialgebraic set.

Theorem 1.1.21 (Schmüdgen's Positivstellensatz, [Sch91]). *Let* $\mathbf{g}$ *be a tuple of polynomials and assume that* $S = \mathcal{S}(\mathbf{g})$ *is compact. Then* $f > 0$ *on* S *implies* $f \in \mathcal{O}(\mathbf{g})$*. In particular, if* S *is compact then* $\mathcal{O}(\mathbf{g})$ *is Archimedean.*

The representation of strictly positive polynomials in the preordering generated by the inequalities is denominator free, but it requires an exponential number of addenda in the number of generators, and this is not a desirable property for applications. A solution to this issue would be a representation of strictly positive polynomials in the quadratic module generated by the inequalities: this representation is provided, in the Archimedeann case, by Putinar's Positivstellensatz.

Theorem 1.1.22 (Putinar's Positivstellensatz, [Put93]). *Let* $\mathbf{g}$ *be a tuple of polynomials and assume that* $\mathcal{Q}(\mathbf{g})$ *is Archimedean. Then* $f > 0$ *on* $\mathcal{S}(\mathbf{g})$ *implies* $f \in \mathcal{Q}(\mathbf{g})$*.*

Theorem 1.1.22 is the basic result behind the convergenec of the Lasserre's hierarchies, see Section 1.6.

We now turn back our attention to vanishing polynomials, and in particular to polynomials vanishing on basic closed semialgebraic sets. Concretely, we want to generalize the Real Nullstellensatz Theorem 1.1.14. For this, we need to introduce the notion of *support*.

Definition 1.1.23. Let Q be a quadratic module. We define the *support* of Q as $\operatorname{supp} Q := Q \cap -Q$.

The support $\operatorname{supp} Q$ can also be defined as the lineality space of the convex cone $Q \subset \mathbb{R}[\mathbf{x}]$, see Section 1.1.1. Notice that, if $Q = \mathcal{Q}(\mathbf{g})$ and $f \in \operatorname{supp} Q = Q \cap -Q$, then $f = 0$ on $\mathcal{S}(\mathbf{g})$. We recall some properties of the support.

Lemma 1.1.24. *Let $Q \subset \mathbb{R}[\mathbf{X}]$ be a quadratic module and $I = \operatorname{supp} Q$. Then:*

(i) I is an ideal;

(ii) $\sqrt{I} = \sqrt[\mathbb{R}]{I}$, i.e. the radical of I is equal to the real radical of I;

(iii) $\dim \frac{\mathbb{R}[\mathbf{X}]}{I} = \dim \frac{\mathbb{R}[\mathbf{X}]}{\sqrt[\mathbb{R}]{I}}$, *where* $\dim$ *denotes the Krull dimension.*

Proof. We briefly prove these known results for the sake of completeness and refer to [Mar08] for more details.

For the first point, closure by addition is trivial. Closure by multiplication follows observing that for all $f \in \mathbb{R}[\mathbf{X}]$ we have $f = (\frac{f+1}{2})^2 - (\frac{f-1}{2})^2 \in \Sigma^2 - \Sigma^2$. Therefore for $f \in \mathbb{R}[\mathbf{x}]$ and $h \in \operatorname{supp} Q = Q \cap -Q$, we have:

$$fh = \Big((\frac{f+1}{2})^2 - (\frac{f-1}{2})^2\Big)h = (\frac{f+1}{2})^2 h - (\frac{f-1}{2})^2 h \in \Sigma^2 \cdot (Q \cap -Q) - \Sigma^2 \cdot (Q \cap -Q) = \operatorname{supp} Q.$$

For the second point, since $\sqrt[\mathbb{R}]{I}$ is a radical ideal we have $\sqrt{I} \subset \sqrt[\mathbb{R}]{I}$. Recall that $\sqrt{I}$ is the intersection of all the minimal prime ideals $\mathfrak{p}$ lying over I. From [Mar08, prop. 2.1.7] we deduce that for such a minimal prime $\mathfrak{p}$, $(Q+\mathfrak{p}) \cap -(Q+\mathfrak{p}) = \mathfrak{p}$ and thus $(\Sigma^2+\mathfrak{p}) \cap -(\Sigma^2+\mathfrak{p}) = \mathfrak{p}$, i.e. $\mathfrak{p}$ is real radical, see Definition 1.1.12. As the intersection of real radical ideals is real radical, we see that $\sqrt{I}$ is a real radical ideal and thus $\sqrt{I} = \sqrt[\mathbb{R}]{I}$.

The last point follows from the second point and the property of the Krull dimension: $\dim \frac{\mathbb{R}[\mathbf{X}]}{I} = \dim \frac{\mathbb{R}[\mathbf{X}]}{\sqrt{I}}$. □

We will then use $\sqrt[\mathbb{R}]{\operatorname{supp} Q}$ to denote both the radical and the real radical of $\operatorname{supp} Q$. The real radical, Definition 1.1.12, can be seen as the radical support of the quadratic module $I + \Sigma^2$. Moreover, one can show that $f \in \sqrt{\operatorname{supp} Q} = \sqrt[\mathbb{R}]{\operatorname{supp} Q}$ if and only if there exists $m \in \mathbb{N}$ and $s \in \Sigma^2$ such that $f^{2m} + s \in \operatorname{supp} Q$ (see discussion after Definition 1.1.12).

We finally use the support to describe the polynomials vanishing on a semialgebraic or algebraic set.

Theorem 1.1.25 (Real Nullstellensatz, [Mar08, note 2.2.2 (vi)])**.** *Let $S = \mathcal{S}(\mathbf{g})$ be a basic closed semialgebraic set. Then for $f \in \mathbb{R}[\mathbf{X}]$, $f = 0$ on S if and only if $f \in \sqrt[\mathbb{R}]{\operatorname{supp} \mathcal{O}(\mathbf{g})}$. In other words, $\mathcal{I}(S) = \sqrt[\mathbb{R}]{\operatorname{supp} \mathcal{O}(\mathbf{g})}$.*

In particular, for an ideal $I \subset \mathbb{R}[\mathbf{X}]$ we have $\mathcal{I}(\mathcal{V}_{\mathbb{R}}(I)) = \sqrt[\mathbb{R}]{I}$.

The preordering $\mathcal{O}(\mathbf{g})$ can be replaced by the quadratic module $\mathcal{Q}(\mathbf{g})$ when the Krull dimension of the quotient $\frac{\mathbb{R}[\mathbf{X}]}{\operatorname{supp} \mathcal{Q}(\mathbf{g})}$ is ≤ 1, as shown in the book of Marshall.

Theorem 1.1.26 ([Mar08, cor. 7.4.2 (3)])**.** *If* $\dim \frac{\mathbb{R}[\mathbf{X}]}{\operatorname{supp} \mathcal{Q}(\mathbf{g})} \leq 1$, *then* $\mathcal{I}(S) = \sqrt[\mathbb{R}]{\operatorname{supp} \mathcal{Q}(\mathbf{g})}$.

We will often use Theorem 1.1.26 in the case $\dim \frac{\mathbb{R}[\mathbf{X}]}{\operatorname{supp} \mathcal{Q}(\mathbf{g})} = 0$.

Truncated quadratic modules

We now introduce *truncated quadratic modules*, convex cones that will play a central role in the thesis.

Definition 1.1.27. For $d \in \mathbb{N}$ and $\mathbf{g}$ a tuple of polynomials, we define the *truncated quadratic module* generated by $\mathbf{g}$ as:

$$\mathcal{Q}_d(\mathbf{g}) := \left\{ s_0 + \sum_{j=1}^{r} s_j g_j \in \mathbb{R}[\mathbf{X}]_d \mid s_i \in \Sigma^2 \ \forall i \in \{0, \dots 1\}, \deg s_0 \leq d, \deg s_j g_j \leq d \ \forall j \in \{1, \dots r\} \right\}.$$

Analogously, we define $\mathcal{O}_d(\mathbf{g}) := \mathcal{Q}_d(\Pi \mathbf{g})$ the *truncated preordering* generated by $\mathbf{g}$.

Notice that $\mathcal{Q}_d(\mathbf{g}) \subset \mathcal{Q}(\mathbf{g}) \cap \mathbb{R}[\mathbf{X}]_d$, but the inclusion is strict in general: a polynomial $f \in \mathcal{Q}(\mathbf{g})$ with $\deg f \leq d$ may need a representation of degree $> d$.

Example 1.1.28. Let $f = x - x^2$. Notice that:

$$x - x^2 = (1-x)^2 x + x^2(1-x).$$

This shows that $f \in \mathcal{Q}(x, 1-x)$, and in particular $f \in \mathcal{Q}_3(x, 1-x)$. But $\deg f = 2 < 3$, and clearly $f \notin \mathcal{Q}_2(x, 1-x)$.

This pathology does not happen for $\Sigma^2 = \mathcal{Q}(1)$: in this case $\Sigma^2 \cap \mathbb{R}[\mathbf{x}]_d = \mathcal{Q}_d(1)$, since the highest degree terms cannot cancel. A generalization of this property is called *stability*.

Definition 1.1.29. A finitely generated quadratic module $\mathcal{Q}(\mathbf{g})$ is called *stable* If for every $d \in \mathbb{N}$ there exists $k = k(d)$ such that $\mathcal{Q}(\mathbf{g}) \cap \mathbb{R}[\mathbf{x}]_d = \mathcal{Q}_k(\mathbf{g}) \cap \mathbb{R}[\mathbf{x}]_d$.

We will be in particular interested in Archimedean quadratic modules $\mathcal{Q}(\mathbf{g})$. Unfortunately, Archimedean quadratic modules are not stable if $\dim \mathcal{S}(\mathbf{g}) \geq 2$ (see [Mar08, p. 149] and [Sch05b, th. 5.4]).

We know from Putinar's Positivstellensatz (Theorem 1.1.22) that a strictly positive polynomial $f > 0$ on $\mathcal{S}(\mathbf{g})$ belongs to $\mathcal{Q}(\mathbf{g})$ in the Archimedean case. The degree needed to represent f as an element of $\mathcal{Q}(\mathbf{g})$ is then a function of several parameters:

- the inequalites $\mathbf{g}$;
- the degree of f;
- the norm of f;
- the minimum f^* of f on $\mathcal{S}(\mathbf{g})$.

The existence of such a degree bound was proven in [PD01, th. 8.3.4], and Chapter 2 will be dedicated to the study of effective versions of this bound. Notice that it is not possible to have a general bound that depends only on $\mathbf{g}$ and $\deg f$, see for instance [Ste96].

1.2 Dualities

In this section, we describe linear functionals acting on polynomials from an algebraic and topological point of view. We first adopt a coordinate-free approach, to emphasize the intrinsic nature of the objects introduced (Section 1.2.1, Section 1.2.3, Section 1.2.4), and then introduce a basis on $\mathbb{K}[\mathbf{x}]$ (namely the monomial basis), and finally the dual basis on $\mathbb{K}[\mathbf{x}]^*$.

1.2.1 Algebraic dual

We come back to the general setting and work over an arbitrary field $\mathbb{K}$. Consider $\mathbb{K}[\mathbf{x}]^* = \hom_{\mathbb{K}}(\mathbb{K}[\mathbf{x}], \mathbb{K}) = \{\Lambda\colon \mathbb{K}[\mathbf{x}] \to \mathbb{K} \mid \Lambda \text{ is } \mathbb{K}\text{-linear}\}$ the algebraic dual space. We denote the application of $\Lambda \in \mathbb{K}[\mathbf{x}]^*$ to $f \in \mathbb{K}[\mathbf{x}]$ by $\langle \Lambda | f \rangle = \Lambda(f)$ to emphasize the duality pairing between $\mathbb{K}[\mathbf{x}]$ and $\mathbb{K}[\mathbf{x}]^*$.

$\mathbb{K}[\mathbf{x}]^*$ is a $\mathbb{K}$-vector space with the usual pointwise addition: $\langle \Lambda_1 + \Lambda_2 | f \rangle = \langle \Lambda_1 | f \rangle + \langle \Lambda_2 | f \rangle$ ($\forall \Lambda_1, \Lambda_2 \in \mathbb{K}[\mathbf{x}]^*$ and $\forall f \in \mathbb{K}[\mathbf{x}]$) and multiplication: $\langle k\Lambda | f \rangle = k \langle \Lambda | f \rangle$ ($\forall \Lambda \in \mathbb{K}[\mathbf{x}]^*$, $\forall f \in \mathbb{K}[\mathbf{x}]$ and $\forall k \in \mathbb{K}$).

For $f \in \mathbb{K}[\mathbf{x}]$, we can consider the multiplication operator $m_f\colon \mathbb{K}[\mathbf{x}] \to \mathbb{K}[\mathbf{x}]$, $g \mapsto fg$. The dual (or transpose) operator defines a structure $\star$ of $\mathbb{K}[\mathbf{x}]$-module on $\mathbb{K}[\mathbf{x}]^*$: for $f \in \mathbb{K}[\mathbf{x}]$ and $\Lambda \in \mathbb{K}[\mathbf{x}]^*$, $f \star \Lambda := \Lambda \circ m_f$, i.e. $\langle f \star \Lambda | g \rangle = \langle \Lambda | fg \rangle$. We denote $\operatorname{Ann}(\Lambda)$ the annihilator of Λ with respect to $\star$, i.e. $\operatorname{Ann}(\Lambda) := \{f \in \mathbb{K}[\mathbf{x}] \mid f \star \Lambda = 0\}$. $\operatorname{Ann}(\Lambda)$ is an ideal in $\mathbb{K}[\mathbf{x}]$.

An important related construction is the so-called *Hankel operator*

$$\begin{aligned} H_\Lambda\colon \mathbb{K}[\mathbf{x}] &\to \mathbb{K}[\mathbf{x}]^* \\ f &\mapsto f \star \Lambda \end{aligned}$$

This is a linear map, and $\ker H_\Lambda = \operatorname{Ann}(\Lambda)$. The bilinear form naturally associated with H_Λ (or with Λ) is:

$$\begin{aligned} \langle \cdot, \cdot \rangle_\Lambda \colon \mathbb{K}[\mathbf{x}] \times \mathbb{K}[\mathbf{x}] &\to \mathbb{K} \\ (f, g) &\mapsto \langle f, g \rangle_\Lambda := \langle H_\Lambda(f) | g \rangle = \langle H_\Lambda(g) | f \rangle = \langle \Lambda | fg \rangle \end{aligned}$$

Among all linear functionals, of special importance are those *induced by a measure* or *coming from a measure*, that we describe hereafter. We assume $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $\mu \in \mathcal{M}(D)$ be a finite Borel measure with support included in $D \subset V$ (see Section 1.3 for the definition, basic properties and references). Consider the map:

$$\begin{aligned} \Lambda_\mu \colon \mathbb{K}[\mathbf{x}] &\to \mathbb{K} \\ f &\mapsto \left\langle \Lambda_\mu \middle| f \right\rangle = \int f \, \mathrm{d}\mu \end{aligned}$$

This is a well-defined linear map on $\mathbb{K}[\mathbf{x}]$ (from the linearity of the integral), and thus $\Lambda_\mu \in \mathbb{K}[\mathbf{x}]^*$. In particular, we are interested in the case where $\mu = \delta_x$ is the Dirac measure: $\int f \, \mathrm{d}\delta_x = f(x)$. We denote $\mathbf{e}_x = \Lambda_{\delta_x}$ the induced linear functional on the polynomial ring:

$$\begin{aligned} \mathbf{e}_x \colon \mathbb{K}[\mathbf{x}] &\to \mathbb{K} \\ f &\mapsto \langle \mathbf{e}_x | f \rangle = \int f \, \mathrm{d}\delta_x = f(x) \end{aligned}$$

We call $\mathbf{e}_x$ the *evaluation* at x. Notice that evaluations are $\mathbb{K}$-algebras homomorphisms, and not only linear maps. Naturally, if $x \in \mathcal{V}_{\mathbb{K}}(I) = X$ for an ideal I, then the evaluation at x can also be seen as an homomorphism $\mathbf{e}_x \colon \mathbb{K}[X] \cong \mathbb{K}[\mathbf{x}]/\mathcal{I}_{\mathbb{K}}(X) \to \mathbb{K}$, see also Section 1.1.4. The maximal ideal associated to x is the kernel of the evaluation at x. We can also construct directly the evaluation linear functional $\langle \mathbf{e}_x | f \rangle = f(x)$, and this does not require the base field $\mathbb{K}$ to be equal to $\mathbb{R}$ or $\mathbb{C}$.

We refer to Section 1.3.2 and Section 1.4 for a more detailed discussion of Borel measures and linear functionals induced by measures.

Orthogonal

Let Z be a $\mathbb{K}$-vector space (we are mainly interested in the case $Z = \mathbb{K}[\mathbf{x}]$ and $Z = \mathbb{K}[\mathbf{x}]_d$). Given a vector subspace $V \subset Z$, one can consider the *orthogonal* of V, namely the linear functionals on Z that vanish on V:

$$V^{\perp} := \{\Lambda \in Z^* \mid \langle \Lambda | v \rangle = 0 \ \forall v \in V\}.$$

This is a vector subspace of Z^*. Moreover, there is a natural isomorphism $V^{\perp} \cong (Z/V)^* = \hom_{\mathbb{K}}(Z/V, \mathbb{K})$.

We are in particular interested in two cases:

- $Z = \mathbb{K}[\mathbf{x}]$ and $V = I$ is an ideal;
- $Z = \mathbb{K}[\mathbf{x}]_d$ and $V = I_d$ or $V = \langle \mathbf{h} \rangle_d$.

Orthogonal spaces of ideals have many interesting properties, especially for zero dimensional ideals: see for instance [EM07].

1.2.2 Moments and pseudo-moments

Before describing in coordinates linear functionals in the dual of $\mathbb{K}[\mathbf{x}]$, we fix $\mathbb{K} = \mathbb{R}$, and consider a Borel measure μ on $\mathbb{R}^n$ (see Section 1.3.2). Then, for $f \in \mathbb{R}[\mathbf{x}]$:

$$\int f \, \mathrm{d}\mu = \int \Big(\sum_{\alpha \in \mathbb{N}^n} f_\alpha \mathbf{x}^\alpha \Big) \mathrm{d}\mu = \sum_{\alpha \in \mathbb{N}^n} f_\alpha \int \mathbf{x}^\alpha \, \mathrm{d}\mu.$$

Therefore, the integration of polynomials with respect to μ is uniquely determined by the values $\mu_\alpha := \int \mathbf{x}^\alpha \, \mathrm{d}\mu$, called *moments* of μ. The sequence $(\mu_\alpha)_{\alpha \in \mathbb{N}^n}$ is called *moment sequence*.

We go back to the case of an arbitrary field $\mathbb{K}$, and generalize the discussion above from measures to linear functionals on the polynomial ring $\mathbb{K}[\mathbf{x}]$. For $\Lambda \in \mathbb{K}[\mathbf{x}]^*$ and $f \in \mathbb{K}[\mathbf{x}]$:

$$\langle \Lambda | f \rangle = \left\langle \Lambda \,\middle|\, \sum_{\alpha \in \mathbb{N}^n} f_\alpha \mathbf{x}^\alpha \right\rangle = \sum_{\alpha \in \mathbb{N}^n} f_\alpha \langle \Lambda | \mathbf{x}^\alpha \rangle.$$

This shows that a linear functional on $\mathbb{K}[\mathbf{x}]$ is uniquely determined from the values $\Lambda_\alpha := \langle \Lambda | \mathbf{x}^\alpha \rangle$, called *pseudo-moments*. The sequence $(\Lambda_\alpha)_{\alpha \in \mathbb{N}^n}$, is therefore called *pseudo-moment sequence*, in analogy with the case of measures. In more abstract terms, the identification $\Lambda \cong (\Lambda_\alpha)_{\alpha \in \mathbb{N}^n}$ above is given by $\mathbb{K}[\mathbf{x}]^* \cong (\bigoplus_{\alpha \in \mathbb{N}^n} \mathbb{K})^* \cong \mathbb{K}^{\mathbb{N}^n}$.

We represent the pseudo-moment sequence $(\Lambda_\alpha)_{\alpha \in \mathbb{N}^n}$ using a generating series, as follows. Let $\Lambda(\mathbf{y}) \in \mathbb{K}[[\mathbf{y}]] := \mathbb{K}[[y_1, \ldots, y_n]]$. Then $\Lambda(\mathbf{y})$ can be written in a unique way as $\Lambda(\mathbf{y}) = \sum_{\alpha \in \mathbb{N}^n} \Lambda_\alpha \frac{\mathbf{y}^\alpha}{\alpha!}$. In topological terms, $(\frac{\mathbf{y}^\alpha}{\alpha!})_{\alpha \in \mathbb{N}^n}$ is a Schauder basis of the topological vector space $\mathbb{K}[[\mathbf{y}]]$. The topology on $\mathbb{K}[[\mathbf{y}]]$ is the $(y_1, \ldots, y_n)$-adic topology, that can be described also as the product topology on $\mathbb{K}[[\mathbf{y}]] \cong \mathbb{K}^{\mathbb{N}^n}$ (where $\mathbb{K}$ is given the discrete topology). We introduce a duality pairing between $\mathbb{K}[\mathbf{x}]$ and $\mathbb{K}[[\mathbf{y}]]$, and interpret $(\frac{\mathbf{y}^\alpha}{\alpha!})_{\alpha \in \mathbb{N}^n}$ as the *dual* basis of $(\mathbf{x}^\alpha)_{\alpha \in \mathbb{N}^n}$: $\left\langle \mathbf{y}^\alpha \middle| \mathbf{x}^\beta \right\rangle = \alpha! \, \delta_{\alpha,\beta}$. In this way, $\langle \Lambda(\mathbf{y}) | f \rangle = \sum_{\alpha \in \mathbb{N}^n} \Lambda_\alpha f_\alpha$, and we have a vector space isomorphism $\mathbb{K}[\mathbf{x}]^* \cong \mathbb{K}[[\mathbf{y}]]$ given by:

$$\begin{aligned} \mathbb{K}[\mathbf{x}]^* &\to \mathbb{K}[[\mathbf{y}]] \\ \Lambda &\mapsto \Lambda(\mathbf{y}) = \sum_{\alpha \in \mathbb{N}^n} \langle \Lambda | \mathbf{x}^\alpha \rangle \frac{\mathbf{y}^\alpha}{\alpha!} = \sum_{\alpha \in \mathbb{N}^n} \Lambda_\alpha \frac{\mathbf{y}^\alpha}{\alpha!} \end{aligned}$$

In particular, $\langle \Lambda | f \rangle = \langle \Lambda(\mathbf{y}) | f \rangle$ for all f and $\Lambda_\alpha = \langle \Lambda | \mathbf{x}^\alpha \rangle$. In the thesis, we will identify $\mathbb{K}[\mathbf{x}]^*$ and $\mathbb{K}[[\mathbf{y}]]$ using the isomorphism above. This formalism has many advantages: we refer to [Mou18] for a comprehensive account of its properties.

We describe now the constructions introduced in Section 1.2.1 using the bases $(\mathbf{x}^\alpha)_{\alpha\in\mathbb{N}^n}$ and $(\frac{\mathbf{y}^\alpha}{\alpha!})_{\alpha\in\mathbb{N}^n}$. If $\Lambda = (\Lambda_\alpha)_{\alpha\in\mathbb{N}^n}$ and $g = \sum_{\beta\in\mathbb{N}^n} g_\beta \mathbf{x}^\beta$, we express $g \star \Lambda$ in the basis $(\frac{\mathbf{y}^\alpha}{\alpha!})_{\alpha\in\mathbb{N}^n}$, computing its pseudo-moments:

$$(g\star\Lambda)_\alpha = \langle g\star\Lambda | \mathbf{x}^\alpha \rangle = \langle \Lambda | \mathbf{x}^\alpha g \rangle = \left\langle \Lambda \middle| \sum_{\beta\in\mathbb{N}^n} g_\beta \mathbf{x}^{\alpha+\beta} \right\rangle = \sum_{\beta\in\mathbb{N}^n} g_\beta \Lambda_{\alpha+\beta}$$

Therefore:

$$(g\star\Lambda)(\mathbf{y}) = \sum_{\alpha\in\mathbb{N}^n} \Big(\sum_{\beta\in\mathbb{N}^n} g_\beta \Lambda_{\alpha+\beta} \Big) \frac{\mathbf{y}^\alpha}{\alpha!}$$

and $(g\star\Lambda)(\mathbf{y})$ is the *cross-correlation* sequence of g and Λ.

We use now the expression of $g\star\Lambda$ as a cross-correlation sequence to describe the Hankel operator $H_\Lambda : \mathbb{K}[\mathbf{x}] \to \mathbb{K}[\mathbf{x}]^*$, $\Lambda \mapsto g\star\Lambda$ using the bases $(\mathbf{x}^\alpha)_{\alpha\in\mathbb{N}^n}$ and $(\frac{\mathbf{y}^\alpha}{\alpha!})_{\alpha\in\mathbb{N}^n}$. The $(\mathbf{x}^\alpha, \frac{\mathbf{y}^\alpha}{\alpha!})$-entry of H_Λ, or equivalently the (α,β)-entry, denoted $(H_\Lambda)_{\alpha,\beta}$, is equal to the pseudo-moment of order β of $H_\Lambda \mathbf{x}^\alpha$. Therefore:

$$(H_\Lambda)_{\alpha,\beta} = (\mathbf{x}^\alpha \star \Lambda)_\beta = \left\langle \mathbf{x}^\alpha \star \Lambda \middle| \mathbf{x}^\beta \right\rangle = \Lambda_{\alpha+\beta}.$$

Then the Hankel operator H_Λ is represented by the so-called *moment matrix* $(\Lambda_{\alpha+\beta})_{\alpha,\beta\in\mathbb{N}^n}$, created from the pseudo-moment sequence $(\Lambda_\alpha)_{\alpha\in\mathbb{N}^n}$.

Remark. Notice that, although the matrix $(\Lambda_{\alpha+\beta})_{\alpha,\beta\in\mathbb{N}^n}$ is called *moment* matrix in the literature, to construct it we only need *pseudo*-moment sequences of linear functionals acting on polynomials, and not moment sequences of measures.

We now investigate the representation of evaluations $\mathbf{e}_x$ using this formalism. We have: $(\mathbf{e}_x)_\alpha = \langle \mathbf{e}_x | \mathbf{x}^\alpha \rangle = x^\alpha$. Therefore:

$$\mathbf{e}_x(\mathbf{y}) = \sum_{\alpha\in\mathbb{N}^n} x^\alpha \frac{\mathbf{y}^\alpha}{\alpha!} = \exp(x\cdot\mathbf{y}),$$

where we used the Taylor expansion of the exponential to express the series in compact form.

Topology

As already mentioned, the formal power series ring comes naturally with the $(y_1,\dots,y_n)$-adic topology. But when $\mathbb{K} = \mathbb{R}$, we can also equip $\mathbb{R}[[\mathbf{y}]]$ with the weak*-topology, see Section 1.2.3: indeed we can equip $\mathbb{R}[\mathbf{x}]$ with the direct limit topology, that is the finest locally convex topology, and $\mathbb{R}[\mathbf{x}]' = \mathbb{R}[\mathbf{x}]^* \cong \mathbb{R}[[\mathbf{y}]]$ can be equipped with the weak*-topology.

A sequence $\Lambda_n(\mathbf{y})$ converge to $\Lambda(\mathbf{y})$ in the $(y_1,\dots,y_n)$-adic topology if and only if for all α, $\Lambda_{n,\alpha} = \Lambda_\alpha$ when $n = n(\alpha)$ is big enough. On the other hand, the convergence in the weak*-topology is the pointwise convergence: $\Lambda_n(\mathbf{y})$ converge to $\Lambda(\mathbf{y})$ in the weak*-topology if and only if $\lim_{n\to\infty} \langle \Lambda_n(\mathbf{y}) | f \rangle = \langle \Lambda(\mathbf{y}) | f \rangle$ for all $f \in \mathbb{R}[\mathbf{x}]$. It is then easy to see that we have weak* convergence if and only $\lim_{n\to\infty} \Lambda_{n,\alpha} = \Lambda_\alpha$ (in the Euclidean topology) for all $\alpha \in \mathbb{N}^n$.

As a concrete simple example, consider the sequence $\Lambda_n(y) = \left(1 + \frac{y}{n}\right)^n$ in the univariate formal power series ring $\mathbb{R}[[y]]$. Then $\Lambda_n(y)$ converges to $\mathbf{e}_1(y) = \exp(y)$ in the weak*-topology, but not in the $(y_1,\ldots,y_n)$-adic topology.

In the thesis, we will equip $\mathbb{R}[[\mathbf{y}]]$ with the weak*-topology. Notice that this topology induces the standard Euclidean topology on every finite dimensional vector subspace $V \subset \mathbb{R}[[\mathbf{y}]]$.

Restriction and truncation

Let $\Lambda \in \mathbb{K}[\mathbf{x}]^*$ or $\Lambda \in (\mathbb{K}[\mathbf{x}]_d)^*$ be a linear functional, and let $k \in \mathbb{N}$ ($k \le d$). We denote $\Lambda^{[k]}$ the restriction $\Lambda\big|_{\mathbb{K}[\mathbf{x}]_k}$ of Λ to $\mathbb{K}[\mathbf{x}]_k$:

$$\left\langle \Lambda^{[k]} \middle| f \right\rangle = \left\langle \Lambda\big|_{\mathbb{K}[\mathbf{x}]_k} \middle| f \right\rangle = \langle \Lambda | f \rangle \text{ for } f \in \mathbb{K}[\mathbf{x}]_k \tag{1.1}$$

We are also interested in the restriction of Hankel operators. For $\Lambda \in \mathbb{K}[\mathbf{x}]^*$ or $\Lambda \in (\mathbb{K}[\mathbf{x}]_d)^*$ ($d \ge 2k$), we define:

$$\begin{aligned} H_\Lambda^k \colon \mathbb{K}[\mathbf{x}]_k &\to \mathbb{K}[\mathbf{x}]_k^* \\ f &\mapsto (f \star \Lambda)^{[k]} \end{aligned}$$

We now describe the operation of restriction of a linear functional in coordinates, in particular restricting to $\mathbb{K}[\mathbf{x}]_k$ for some $k \in \mathbb{N}$.

Let $\Lambda \in \mathbb{K}[\mathbf{x}]^*$ or $\Lambda \in (\mathbb{K}[\mathbf{x}]_d)^*$ be a linear functional, and let $k \in \mathbb{N}$ ($k \le d$). Then $\Lambda^{[k]}$ is uniquely determined by its application to polynomials of degree $\le k$, and thus it is uniquely determined by its sequence of pseudo-moments of degree $\le k$: $\Lambda^{[k]} = (\Lambda_\alpha)_{|\alpha| \le k}$. Analogously, the truncated moment matrix has the following expression ($2k \le d$):

$$[H_\Lambda^k] = (\Lambda_{\alpha+\beta})_{|\alpha| \le k, |\beta| \le k}.$$

1.2.3 Topological dual

In this section we consider $\mathbb{K} = \mathbb{R}$ the field of real numbers. We briefly recall properties of topological vector spaces and their duals, and refer to [Bar02; Rud91] for more details.

A *topological vector space* (Z, τ) is an $\mathbb{R}$-vector space Z where addition and multiplication by scalars are continuous with respect to the topology τ of Z, and such that points of Z are cloded for τ. This implies that the space is Hausdorff, see [Rud91, th. 1.12]. We will omit to specify the topology τ when it is clear from the context. For a topological vector space Z over $\mathbb{R}$, denote Z' the vector space of *continuous* linear functionals on Z, i.e. $Z' := \{\Lambda \colon Z \to \mathbb{R} \mid \Lambda \text{ is linear and continuous}\}$. We are in particular interested in the case $Z = \mathbb{R}[\mathbf{x}]$. We need first to introduce a topology on it. Notice that $\mathbb{R}[\mathbf{x}]$ is a countably dimensional vector space over $\mathbb{R}$, since the monomials $\{\mathbf{x}^\alpha\}_{\alpha \in \mathbb{N}^n}$ form a basis.

Let Z be a countable dimensional vector space over $\mathbb{R}$. We can equip Z with the *direct limit topology*, defined as follows. Equip every finite dimensional subspace $W \subset Z$ with the Euclidean topology. Then $U \subset Z$ is open in Z if and only if $U \cap W$ is open in W. This topology is *locally convex* (when Z is countable dimensional): the open sets that are convex form a basis for the topology. Addition and multiplication by scalars are continuous with respect to the direct limit topology, therefore Z has the structure of topological vector space.

Any locally convex topology can be described as the coarsest topology such that a family of *seminorms* on Z is continuous (a map $p\colon V \to \mathbb{R}_{\geq 0}$ is called a seminorm if $\langle p|xv\rangle = |x|\langle p|v\rangle$ and $\langle p|v+w\rangle \leq \langle p|v\rangle + \langle p|w\rangle$ for all $x \in \mathbb{R}$ and $v, w \in Z$). The direct limit topology on a cuntable dimensional vector space is the *finest* locally convex topology, or in other words it is the coarsest topology such that *all* seminorms on V are continuous. Every linear functional is continuous on Z with respect to this topology: therefore the topological and algebraic dual coincide (as vector spaces): $Z' = Z^*$.

Since the $\mathbb{R}[\mathbf{x}]$ is countably dimensional, we can equip it with the direct limit topology, that coincides with the finest locally convex topology.

Now, we want to give a topology on the dual space Z': the weak*-topology. The weak*-topology is defined as the coarsest topology on Z' such that every $v \in Z$ is continuous (v is seen as a linear functional $v\colon \Lambda \mapsto \langle \Lambda|v\rangle = \Lambda(v)$ on Z). The weak*-topology gives Z' the structure of locally convex topological vector space, and $(Z')' \cong Z$: any $v \in Z$ defines a linear functional $v\colon Z' \to \mathbb{R}$, that is continuous by definition of weak*-topology, and moreover every linear continuous linear functional on Z' has this form, see [Rud91, sec. 3.14].

We consider in particular the case $Z = \mathbb{R}[\mathbf{x}]$. Notice that $\mathbb{R}[\mathbf{x}] \cong \bigoplus_{\mathbb{N}^n} \mathbb{R}$ (as a vector space, for instance choosing the standard monomial basis), and $\mathbb{R}[\mathbf{x}]^* \cong (\bigoplus_{\mathbb{N}^n} \mathbb{R})^* \cong \prod_{\mathbb{N}^n} \mathbb{R} = \mathbb{R}^{\mathbb{N}^n}$. We equip $\mathbb{R}[\mathbf{x}]^*$ with the weak*-topology, and therefore $(\mathbb{R}[\mathbf{x}]')' = (\mathbb{R}[\mathbf{x}]^*)' \cong \mathbb{R}[\mathbf{x}]$.

1.2.4 Convex duality

In Section 1.2.3 we fixed the notations and definition needed to consider duals of topological vector spaces. Hereafter we state the main results of convex duality that we will need through the article.

The first result is the Separation Theorem for convex sets. It is the key result to describe convex sets using linear functionals or, equivalently, affine halfspaces.

Theorem 1.2.1 (Separation Theorem, [Rud91, th. 3.4])**.** *Let A and B be disjoint, nonempty convex sets in a topological vector space Z.*

(i) If A is open there exists $\Lambda \in Z'$ and $c \in \mathbb{R}$ such that $\langle \Lambda|a\rangle < c$ for all $a \in A$ and $\langle \Lambda|b\rangle > c$ for all $b \in B$;

(ii) If Z is locally convex, A is compact and B is closed, then there exist $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, and $\Lambda \in Z'$ such that $\langle \Lambda|a\rangle < r_1$ for all $a \in A$ and $\langle \Lambda|b\rangle > r_2$ for all $b \in B$.

Definition 1.2.2 (Dual Cone)**.** Let $C \subset Z$ be a convex cone in a topological vector space Z. The dual cone $C^\vee$ is defined as

$$C^\vee := \{\Lambda \in Z' \mid \langle \Lambda|c\rangle \geq 0 \text{ for all } c \in C\}$$

We are interested in the case when Z is locally convex and we equip Z' with the weak*-topology (that is a locally convex topology). In this case we have $(Z')' \cong Z$ (see Section 1.2.3), and we can therefore consider the double dual in Z:

$$(C^\vee)^\vee = \{v \in Z \mid \langle \Lambda|v\rangle \geq 0 \text{ for all } \Lambda \in C^\vee\}$$

Corollary 1.2.3 (Conic Duality)**.** *Let C be a convex cone in a locally convex topological vector space Z. Then $\overline{C} = (C^\vee)^\vee$.*

Proof. For $c \in C$ and $\Lambda \in C^\vee$ we have $\langle \Lambda | c \rangle \geq 0$ from the definition of $C^\vee$, and this shows that $C \subset (C^\vee)^\vee$. Since $(C^\vee)^\vee$ is closed, we have also $\overline{C} \subset (C^\vee)^\vee$.

Now, assume that $\overline{C} \subsetneq (C^\vee)^\vee$ and pick $c \in (C^\vee)^\vee \setminus \overline{C}$. We apply Theorem 1.2.1 to $\{c\}$, that is compact, and $\overline{C}$: there exists a linear functional Λ and $r_1 < r_2 \in \mathbb{R}$ such that $\langle \Lambda | c \rangle < r_1$ and $\langle \Lambda | v \rangle > r_2$ for all $v \in \overline{C}$. In particular, since $\overline{C}$ is a convex cone, $\langle \Lambda | rv \rangle \geq r_2$ for all $v \in \overline{C}$ and $r \in \mathbb{R}_{\geq 0}$: therefore $\langle \Lambda | v \rangle \geq \frac{r_2}{r}$ for all $r \in \mathbb{R}_{\geq 0}$. This implies $\langle \Lambda | v \rangle \geq 0$ for all $v \in \overline{C}$, i.e. $\Lambda \in C^\vee$, and that $r_2 \leq 0$. But this is a contradiction: indeed we have $\langle \Lambda | c \rangle \geq 0$ as $c \in (C^\vee)^\vee$ and $\langle \Lambda | c \rangle < r_1 < r_2 \leq 0$. Therefore, $\overline{C} = (C^\vee)^\vee$. □

Another important result is the Kreil-Milman's Theorem, describing the convex hull of a compact set using extreme points.

Theorem 1.2.4 (Kreil-Milman's Theorem, [Bar02, th. 4.1])**.** *Let Z be a locally convex topological vector space and let $K \subset Z$ be a compact convex set. Then K is equal to the closure of the convex hull of its extreme points.*

Finally, we recall the Minkowski theorem, describing the convex hull using extreme points in the finite dimensional case.

Theorem 1.2.5 (Minkowski's Theorem, [Bar02, th. 3.3])**.** *Let Z a finite dimensional real vector space equipped with the Euclidean topology, and let $K \subset Z$ be a compact set. Then K is equal to the convex hull of its extreme points.*

1.3 A Bestiary of convex cones and their duals

Hereafter we describe cones that will be frequently used in the thesis. We define new cones, recall the definition of others already introduced and briefly summarize all the properties that will be used in the thesis.

1.3.1 Positive polynomials

The convex cone of polynomials positive on a domain D has been introduced in Section 1.1.6:

$$\mathrm{Pos}(D) = \{f \in \mathbb{R}[\mathbf{x}] \mid f(x) \geq 0 \text{ for all } x \in D\}.$$

Clearly, $\mathrm{Pos}(D) = \overline{\mathrm{Pos}(D)}$ is a closed convex cone (as a subset of $\mathbb{R}[\mathbf{x}]$ with the locally convex topology): indeed $\mathrm{Pos}(D) \cap \mathbb{R}[\mathbf{x}]_d$ is closed for all $d \in \mathbb{N}$, since the limit of a converging sequence of positive polynomials on D of bounded degree is positive on D. When $D = \mathcal{S}(\mathbf{g})$ is a semialgebraic set, a complete description of $\mathrm{Pos}(\mathcal{S}(\mathbf{g}))$ as ratio of polynomials in the preordering $\mathcal{O}(\mathbf{g})$ is given by the Krivine-Stengle Positivstellensatz, Theorem 1.1.18.

1.3.2 Borel measures

We give a simple introduction to Borel measures on $\mathbb{R}^n$, following [Lau09; Mar08]. Let $\mathcal{B}(\mathbb{R}^n)$ denote the Borel σ-algebra generated by the open subsets of $\mathbb{R}^n$ (or, equivalently, generated by the compact subsets of $\mathbb{R}^n$). A *Borel measure* is a positive measure μ on $\mathcal{B}(\mathbb{R}^n)$, i.e. $\mu \colon \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ such that $\mu(\emptyset) = 0$, and $\mu(\bigcup_{i \in \mathbb{N}} A_i) = \sum_{i \in \mathbb{N}} \mu(A_i)$ for any pairwise disjoint $A_i \in \mathcal{B}(\mathbb{R}^n)$. We say that a Borel measure is *regular* if, for all $A \in \mathcal{B}(X)$, $\mu(A)$ can be approximated from below by $\mu(K)$, where $K \subset A$ is compact, and if it can be approximated from above by $\mu(U)$, where $U \supset A$ is open. If $\mu(A) < +\infty$ for all $\mu \in \mathcal{M}(K)$ and $A \in \mathcal{B}(\mathbb{R}^n)$, we say μ is *finite*.

The *support* of $\mu \in \mathcal{M}(\mathbb{R}^n)$, denoted $\mathrm{supp}(\mu)$, is the smallest closed set $S \subset \mathbb{R}^n$ such that $\mu(\mathbb{R}^n \setminus S) = 0$. We denote $\mathcal{M}(D)$ the (regular, finite) Borel measures μ such that $\mathrm{supp}(\mu) \subset D \subset \mathbb{R}^n$, and by $\mathcal{M}^{(1)}(D)$ the *probability* measures, i.e. the $\mu \in \mathcal{M}(D)$ such that $\mu(D) = \mu(\mathbb{R}^n) = \int 1 \, \mathrm{d}\mu = 1$.

We are interested in particular in the case $D = K$ is compact. Notice that in this case any $\mu \in \mathcal{M}(D)$ is finite.

Using Riesz's Representation theorem (see for instance [Sch17, th. A.4] or [Mar08, th. 3.2.1] and references therein) one can identify $\mathcal{M}(K)$ as the dual cone of positive continuous functions on K.

As we are interested in polynomials (and not on continuous functions), we need to consider (finite) Borel measures acting only on polynomials. Formally, consider

$$i \colon \mathcal{M}(D) \to \mathbb{R}[\mathbf{x}]^*, \ \mu \mapsto \Lambda_\mu,$$

where $\langle \Lambda_\mu | f \rangle = \int f \, \mathrm{d}\mu$ as in Section 1.2.1. If $D = K$ is compact, then i is injective as a consequence of Stone-Weierstrass Approximation theorem (this is the *determinacy* of the moment problem, see also Section 1.4). Using the indentification above, with an abuse of notation we write $\mathcal{M}(K) \subset \mathbb{R}[\mathbf{x}]^*$.

Then there is a natural question: can we describe $\mathcal{M}(K) \subset \mathbb{R}[\mathbf{x}]^*$ as the dual cone of positive polynomials? That is, can we replace positive continuous functions with positive polynomials? The positive answer is given by Haviland's theorem (see [Mar08, th. 3.1.2] or [Sch17, th. 1.12] and references therein).

Theorem 1.3.1 (Haviland's Theorem)**.** *Given $\Lambda \in \mathbb{R}[\mathbf{x}]$ and $D \subset \mathbb{R}^n$, the following are equivalent:*

(i) $\langle \Lambda | f \rangle \geq 0$ *for all* $f \in \mathrm{Pos}(D)$*;*

(ii) $\Lambda = \Lambda_\mu$ *for some* $\mu \in \mathcal{M}(D)$*.*

In particular, if $D = K$ is compact, then $\mathcal{M}(K) = \mathrm{Pos}(K)^\vee \subset \mathbb{R}[\mathbf{x}]^*$.

Furthermore, notice that $\mathcal{M}(K) = \mathrm{cone}(\mathcal{M}^{(1)}(K))$. One is naturally interested in extremal point of $\mathcal{M}^{(1)}(K)$, or equivalently in extremal rays of $\mathcal{M}(K)$. These extremal rays are precisely the Dirac measures δ_x or evaluations $\mathbf{e}_x$ at points of K, see for instance [Bar02, prop. (8.4)]. Therefore, from the Krein-Milman theorem (Theorem 1.2.4) we obtain $\mathcal{M}(K) = \overline{\mathrm{cone}(\{\mathbf{e}_x \colon x \in K\})}$.

1.3.3 Positive semidefinite matrices

We introduce positive semidefinite matrices, a convex cone in the space of real symmetric matrices. We refer to [Bar02, ch. II, sec. 12] and [BPT12, A.3.5] for proofs and more details.

Proposition 1.3.2. *Let M be an $n \times n$ symmetric matrix. The following are equivalent:*

- $x^t M x \geq 0$ *for all* $x \in \mathbb{R}^n$*;*
- *All the eigenvalues of M are* ≥ 0*;*

- *$M = U^tU$ for some $n \times n$ matrix U (Gram decomposition);*
- *All the determinants of the principal minors of M are ≥ 0;*
- *M is a convex combination of matrices of the form xx^t, with $x \in \mathbb{R}^n$.*

If a symmetric matrix M satisfies any of the above equivalent conditions, M is called *positive semidefinite* and we write $M \succcurlyeq 0$. If $x^tMx > 0$ for all $0 \neq x \in \mathbb{R}^n$ (or if all the eigenvalues of M are > 0, or if $M = U^tU$ for some $n \times n$, rank n matrix U) then the symmetric matrix M is called *positive definite*, and we write $M \succ 0$.

If we equip the space $\mathcal{S}^n$ of symmetric matrices with the trace inner product $\langle M, N\rangle = \operatorname{tr}(MN)$ to identify $(\mathcal{S}^n)^*$ with $\mathcal{S}^n$, then the convex cone of positive semidefinite matrices $\mathcal{S}^n_+$ is self dual. Positive definite matrices are the interior points of $\mathcal{S}^n_+$.

The cone of positive definite matrices is important, since the affine slices of this cone are the feasible regions of *semidefinite programming*. These affine slices of $\mathcal{S}^n_+$ are called *spectrahedra*, and are convex sets. Their projections are called *spectrahedral shadows*. An important, nontrivial result is that there exist convex cones that are not spectrahedral shadows, see [Sch18].

1.3.4 Sums of squares

We review properties of convex cone of Sums of Squares polynomials, that lives in the polynomial ring $\mathbb{R}[\mathbf{x}]$ equipped with the locally convex topology.

Recall the definition of Sums of Squares polynomials:

$$\Sigma^2 = \Sigma\mathbb{R}[\mathbf{x}]^2 = \left\{ f \in \mathbb{R}[\mathbf{X}] \mid \exists r \in \mathbb{N},\ g_i \in \mathbb{R}[\mathbf{X}] \colon f = g_1^2 + \cdots + g_r^2 \right\}.$$

This is a closed, pointed, full dimensional cone, and it is stable (as a quadratic module $\mathcal{Q}(1) = \Sigma^2$, see Definition 1.1.27). We briefly present now the deep connection between SoS polynomials and positive semidefinite matrices, see for instance [Lau09, sec. 3.3] or [Mar08, lem. 4.1.3] and references therein.

Let $f = \sum_{i=1}^r h_i^2 \in \Sigma^2$, $\deg f \leq 2d$, and let $\mathbf{b}_d = b_{d,1}, b_{d,2}, \ldots$ be a basis of $\mathbb{R}[\mathbf{x}]_d$ (for instance, $\mathbf{b}_d$ can be chosen to be the monomial basis, see Section 1.1.3). Since the highest degree coefficients of $h_1^2, \ldots, h_r^2$ cannot cancel we have $\deg h_i^2 \leq 2d$ for all i. Write $h_i = \mathbf{b}_d^t \operatorname{vec}(h_i) = \sum_j h_{i,j} b_j$. Then, if $\mathbf{b}_{2d} = b_{2d,1}, b_{2d,2}, \ldots$ is a basis of $\mathbb{R}[\mathbf{x}]_{2d}$, we have:

$$h_i^2 = \mathbf{b}_d^t \operatorname{vec}(h_i)\operatorname{vec}(h_i)^t \mathbf{b}_d = \sum_\ell \Big(\sum_{\substack{j,k: \\ b_{d,j}b_{d,k} = b_{2d,\ell}}} h_{i,j}h_{i,k} \Big) b_{2d,\ell} = \mathbf{b}_d^t H_i \mathbf{b}_d \tag{1.2}$$

$$f = \sum_i \mathbf{b}_d^t H_i \mathbf{b}_d = \sum_\ell \Big(\sum_i \sum_{\substack{j,k: \\ b_{d,j}b_{d,k} = b_{2d,\ell}}} h_{i,j}h_{i,k} \Big) b_{2d,\ell} = \mathbf{b}_d^t A \mathbf{b}_d \tag{1.3}$$

where $A = \sum_i H_i$ and H_i has (j,k)-entry equal to $h_{i,j}h_{i,k}$. For every i, $h_i^2 \geq 0$ on $\mathbb{R}^n$, then H_i is positive semidefinite and finally A is positive semidefinite. On the other hand if $A = U^tU$ is positive semidefinite, then, denoting u_i the rows of U:

$$f := \mathbf{b}_d^t A \mathbf{b}_d = (U\mathbf{b}_d)^t(U\mathbf{b}_d) = ((u_i \cdot \mathbf{b}_d)_i)^t (u_i \cdot \mathbf{b}_d)_i = \sum_i (u_i \cdot \mathbf{b}_d)^2 = \sum_i h_i^2,$$

and f is a SoS polynomial.

As a consequence, equating the coefficients with the semidefinite representation we see that $f = \sum_\ell f_\ell b_{2d,\ell}$ is a SoS polynomial if and only if the following semidefinite program is feasible:

$$\sum_{\substack{j,k:\\ b_{d,j}b_{d,k}=b_{2d,\ell}}} A_{j,k} = f_\ell \ \forall \ell\colon\ b_{2d,\ell} \in \mathbf{b}_{2d}, \quad A = (A_{j,k}) \in \mathcal{S}_+^{|\mathbf{b}_d|}.$$

The solution set of the semidefinite program above is called *Gram spectrahedron* of f.

We now more explicitly express the semidefinite program above using symmetric matrices and the trace inner product on $\mathcal{S}^{|\mathbf{b}_d|}$. f can be seen as a 1×1 matrix with entries in $\mathbb{R}[\mathbf{x}]_{2d}$. Therefore, from (1.3):

$$f = \operatorname{tr}(f) = \operatorname{tr}\left(\mathbf{b}_d^t A \mathbf{b}_d\right) = \operatorname{tr}\left(A\mathbf{b}_d\mathbf{b}_d^t\right) = \operatorname{tr}(AC_d) = \langle A, C_d\rangle,$$

using the cyclic property of the trace and setting $C_d := \mathbf{b}_d\mathbf{b}_d^t = (b_{d,j}b_{d,k}) \in \mathcal{S}^{|\mathbf{b}_d|}$. C_d is a symmetric matrix with polynomial antries, or equivalently can be seen as a polynomial with matrix coefficients. We write then $C_d = \sum_{\ell\colon b_{2d,\ell}\in\mathbf{b}_{2d}} b_{2d,\ell}C_{d,\ell}$, where $C_{d,\ell}$ is the matrix with (j,k)-entry equal to the coefficient of $b_{2d,\ell}$ in the (j,k)-entry of C_d. Therefore we obtain that $f \in \Sigma^2$ if and only if the following semidefinite program is feasible:

$$\langle A, C_{d,\ell}\rangle = f_\ell \ \forall \ell\colon\ b_{2d,\ell} \in \mathbf{b}_{2d}, \quad A = (A_{j,k}) \in \mathcal{S}_+^{|\mathbf{b}_d|}.$$

Example 1.3.3. Let $f = (x-1)^2 + y^2 = 1 - 2x + x^2 + y^2 \in \Sigma^2 \subset \mathbb{R}[x,y]_2$, and we choose the monomial basis. Notice that:

$$(x-1)^2 = 1 - 2x + x^2 = \begin{pmatrix}1 & x & y\end{pmatrix}\begin{pmatrix}1 & -1 & 0\\ -1 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\ x\\ y\end{pmatrix}$$

$$y^2 = \begin{pmatrix}1 & x & y\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\ x\\ y\end{pmatrix}$$

Therefore:

$$f = \begin{pmatrix}1 & x & y\end{pmatrix}\left(\begin{pmatrix}1 & -1 & 0\\ -1 & 1 & 0\\ 0 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\right)\begin{pmatrix}1\\ x\\ y\end{pmatrix} = \begin{pmatrix}1 & x & y\end{pmatrix}\begin{pmatrix}1 & -1 & 0\\ -1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\ x\\ y\end{pmatrix}$$

Now notice that:

$$\begin{aligned} f = \operatorname{tr}(f) &= \operatorname{tr}\left(\begin{pmatrix}1 & x & y\end{pmatrix}\begin{pmatrix}1 & -1 & 0\\ -1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\ x\\ y\end{pmatrix}\right) = \operatorname{tr}\left(\begin{pmatrix}1 & -1 & 0\\ -1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\ x\\ y\end{pmatrix}\begin{pmatrix}1 & x & y\end{pmatrix}\right)\\ &= \operatorname{tr}\left(\begin{pmatrix}1 & -1 & 0\\ -1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & x & y\\ x & x^2 & xy\\ y & xy & y^2\end{pmatrix}\right)\end{aligned}$$

Write:

$$\begin{pmatrix} 1 & x & y \\ x & x^2 & xy \\ y & xy & y^2 \end{pmatrix} = 1\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
$$= x^2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + xy\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + y^2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

And finally:

$$\begin{cases} f_{(0,0)} = 1 = \operatorname{tr}\left(\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \\ f_{(1,0)} = -2 = \operatorname{tr}\left(\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \\ f_{(0,1)} = 0 = \operatorname{tr}\left(\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\right) \\ f_{(2,0)} = 1 = \operatorname{tr}\left(\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \\ f_{(1,1)} = 0 = \operatorname{tr}\left(\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right) \\ f_{(0,2)} = 1 = \operatorname{tr}\left(\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \end{cases}$$

1.3.5 Quadratic modules

We now generalize from SoS to quadratic modules (in particular, finitely generated ones). Given a tuple $\mathbf{g}$ of r polynomials, recall the definition:

$$\mathcal{Q}(\mathbf{g}) = \mathcal{Q}(g_1, \dots, g_r) = \Sigma^2 + \Sigma^2 \cdot g_1 + \cdots + \Sigma^2 \cdot g_r.$$

With this notation, $\Sigma^2 = \mathcal{Q}(1)$. Notice that $\mathcal{Q}(\mathbf{g}) - \mathcal{Q}(\mathbf{g}) = \mathbb{R}[\mathbf{x}]$, since $\Sigma^2 \subset \mathcal{Q}(\mathbf{g})$. Moreover, if $\mathcal{S}(\mathbf{g}) \subset \mathbb{R}^n$ is full dimensional (i.e. $\mathcal{S}(\mathbf{g})$ contains an open ball in the euclidean topology) then $\operatorname{supp} \mathcal{Q}(\mathbf{g}) = \mathcal{Q}(\mathbf{g}) \cap -\mathcal{Q}(\mathbf{g}) = (0)$. The converse is not true, see for instance Example 3.3.12.

In general, quadratic modules are not closed, and $\mathcal{Q}(\mathbf{g}) \subset \operatorname{Pos}(\mathcal{S}(\mathbf{g}))$. Studying closures of arbitrary finitely generated quadratic modules is hard: the main results in this investigation are the following.

- If $\mathcal{Q}(\mathbf{g})$ is Archimedean, then $\overline{\mathcal{Q}(\mathbf{g})} = \operatorname{Pos}(\mathcal{S}(\mathbf{g}))$. Indeed, if $f \in \operatorname{Pos}(\mathcal{S}(\mathbf{g}))$ then from Putinar's Positivstellensatz (Theorem 1.1.22) $f + \varepsilon \in \mathcal{Q}(\mathbf{g})$ for all $\varepsilon > 0$, and thus $f \in \overline{\mathcal{Q}(\mathbf{g})}$.

- If $\mathcal{Q}(\mathbf{g})$ is stable, then $\overline{\mathcal{Q}(\mathbf{g})} = \mathcal{Q}(\mathbf{g}) + \sqrt[\mathbb{R}]{\operatorname{supp}\mathcal{Q}(\mathbf{g})}$, see [Sch05b, th. 3.17].

We present two explicit examples where the quadratic module $\mathcal{Q}(\mathbf{g})$ is not closed.

Example 1.3.4. Let $Q = \mathcal{Q}(-x^2) \subset \mathbb{R}[x]$. Since Q is Archimedean then $\overline{Q} = \operatorname{Pos}(\{0\})$. In particular, for all $\varepsilon > 0$:

$$x + \frac{\varepsilon}{2} = \frac{\varepsilon}{2}\Big(\Big(1 - \frac{x}{\varepsilon}\Big)^2 - \frac{x^2}{\varepsilon^2}\Big) \in \mathcal{Q}_2(-x^2) \subset Q$$

Since clearly $x \notin Q$, we have $x \in \operatorname{Pos}(\{0\}) \setminus Q = \overline{Q} \setminus Q$ and Q is not closed. Furthermore, this example also shows that $\mathcal{Q}_2(-x^2)$ is not closed. This happens because $\operatorname{supp} Q = (x^2)$ is not radical: see Theorem 3.4.3.

Example 1.3.5. Let $Q = \mathcal{Q}(x^3 - y^2, 1 - x^2 - y^2)$. In this case $x \notin Q$, since $x \notin \mathcal{Q}(x^3, 1 - x^2) \subset \mathbb{R}[x]$, but $x \in \operatorname{Pos}(\mathcal{S}(x^3 - y^2, 1 - x^2 - y^2)) = \overline{Q}$. Therefore, Q is not closed. See also Example 3.5.16.

1.3.6 Positive linear functionals

We are interested in cones that are dual to quadratic modules. Given a quadratic module Q, we define the convex cone of *Posisite Linear Functionals* on Q as:

$$\mathcal{L}(Q) := Q^\vee = \{\Lambda \in \mathbb{R}[\mathbf{x}]^\vee \mid \langle \Lambda | q \rangle \geq 0 \ \forall q \in Q\}.$$

When $Q = \mathcal{Q}(\mathbf{g})$ is finitely generated, we write $\mathcal{L}(\mathbf{g}) := \mathcal{L}(Q)$. More generally, if $G \subset \mathbb{R}[\mathbf{x}]$ and $\mathcal{Q}(G)$ denotes the smallest quadratic module containing G, we denote $\mathcal{L}(G) := \mathcal{L}(\mathcal{Q}(G))$. In particular, if $I \subset \mathbb{R}[\mathbf{x}]$ is an ideal, then

$$\mathcal{L}(I) = (I + \Sigma^2)^\vee = \{\Lambda \in I^\perp \mid \langle \Lambda | s \rangle \geq 0 \ \forall s \in \Sigma^2\}.$$

Since $\operatorname{Pos}(\mathcal{S}(\mathbf{g})) \supset \mathcal{Q}(\mathbf{g})$, dualizing we obtain $\mathcal{M}(\mathcal{S}(\mathbf{g})) \subset \mathcal{L}(\mathbf{g}) = \mathcal{L}(\mathcal{Q}(\mathbf{g}))$. Whether this inclusion is an equality or not is part of the study of the *Moment Problem*, see Section 1.4. The positive solution to this question was obtained for preorderings definying a compact semialgebraic set by Schmüdgen and for Archimedean quadratic modules by Putinar.

Theorem 1.3.6 ([Sch91]). *Let $\mathbf{g}$ be a tuple of polynomials and assume that $S = \mathcal{S}(\mathbf{g})$ is compact. Then, if $\Lambda \in \mathbb{R}[\mathbf{x}]^*$ is such that $\langle \Lambda | p \rangle \geq 0$ for all $p \in \mathcal{O}(\mathbf{g})$, there exists $\mu \in \mathcal{M}(\mathcal{S}(\mathbf{g}))$ such that $\Lambda = \Lambda_\mu$, and $\mathcal{M}(\mathcal{S}(\mathbf{g})) = \mathcal{O}(\mathbf{g})^\vee = \mathcal{L}(\Pi\mathbf{g})$.*

Theorem 1.3.7 ([Put93]). *Let $\mathbf{g}$ be a tuple of polynomials and assume that $\mathcal{Q}(\mathbf{g})$ is Archimedean. Then, if $\Lambda \in \mathbb{R}[\mathbf{x}]^*$ is such that $\langle \Lambda | q \rangle \geq 0$ for all $q \in \mathcal{O}(\mathbf{g})$, there exists $\mu \in \mathcal{M}(\mathcal{S}(\mathbf{g}))$ such that $\Lambda = \Lambda_\mu$, and $\mathcal{M}(\mathcal{S}(\mathbf{g})) = \mathcal{Q}(\mathbf{g})^\vee = \mathcal{L}(\mathbf{g})$.*

Originally, Theorem 1.1.21 and Theorem 1.1.22 were proven as corollaries of Theorem 1.3.6 and Theorem 1.3.7. These results are important, since finite generated preorderings and quadratic modules are much simpler than the cone of positive polynomials.

We now describe elements of $\mathcal{L}(\mathbf{g})$ using an infinite dimensional analogue of positive semidefinite matrices. Notice that:

$$\begin{aligned}\Lambda \in \mathcal{L}(\mathbf{g}) &\iff \langle \Lambda | q\rangle \geq 0 \ \forall q \in \mathcal{Q}(\mathbf{g})\\ &\iff \langle \Lambda | s\rangle \geq 0, \langle \Lambda | s g_1\rangle \geq 0, \dots, \langle \Lambda | s g_r\rangle \geq 0 \ \forall s \in \Sigma^2\\ &\iff \left\langle \Lambda \middle| h^2\right\rangle \geq 0, \left\langle \Lambda \middle| h^2 g_1\right\rangle \geq 0, \dots, \left\langle \Lambda \middle| h^2 g_r\right\rangle \geq 0 \ \forall h \in \mathbb{R}[\mathbf{x}]\\ &\iff \left\langle \Lambda \middle| h^2\right\rangle \geq 0, \left\langle g_1 \star \Lambda \middle| h^2\right\rangle \geq 0, \dots, \left\langle g_r \star \Lambda \middle| h^2\right\rangle \geq 0 \ \forall h \in \mathbb{R}[\mathbf{x}]\\ &\iff \langle h \star \Lambda | h\rangle \geq 0, \langle h \star (g_1 \star \Lambda) | h\rangle \geq 0, \dots, \langle h \star (g_r \star \Lambda) | h\rangle \geq 0 \ \forall h \in \mathbb{R}[\mathbf{x}].\end{aligned}$$

Recall the definition of Hankel operator, see Section 1.2.1: $H_\Lambda(h) = h \star \Lambda$, and the associated quadratic form on $\mathbb{R}[\mathbf{x}]$ given by $\langle h, h\rangle_\Lambda = \langle H_\Lambda(h)|h\rangle = \langle h \star \Lambda | h\rangle = \left\langle \Lambda \middle| h^2\right\rangle$ for $h \in \mathbb{R}[\mathbf{x}]$. As is the finite dimensional case, we say that H_Λ is *positive semidefinte* if $\langle H_\Lambda(h)|h\rangle = \langle h \star \Lambda | h\rangle \geq 0$ for all $h \in \mathbb{R}[\mathbf{x}]$, and we write $H_\Lambda \succcurlyeq 0$. Therefore:

$$\mathcal{L}(\mathbf{g}) = \{\Lambda \in \mathbb{R}[\mathbf{x}]^* \mid H_\Lambda \succcurlyeq 0, H_{g_1 \star \Lambda} \succcurlyeq 0, \dots, H_{g_r \star \Lambda} \succcurlyeq 0\}.$$

Finally, notice that if $H_\Lambda \succcurlyeq 0$ (and in particular, if $\Lambda \in \mathcal{L}(\mathbf{g})$), then $\langle \Lambda | 1\rangle = 0$ implies $\Lambda = 0$. Indeed, if $\langle \Lambda | 1\rangle = 0$ and $h \in \mathbb{R}[\mathbf{x}]$, then choosing a basis of $\mathbb{R}[\mathbf{x}]$ containing 1 and h, the minor of H_Λ indexed by 1 and h has form:

$$\begin{pmatrix} \langle \Lambda | 1\rangle & \langle \Lambda | h\rangle \\ \langle \Lambda | h\rangle & \left\langle \Lambda \middle| h^2\right\rangle \end{pmatrix} = \begin{pmatrix} 0 & \langle \Lambda | h\rangle \\ \langle \Lambda | h\rangle & \left\langle \Lambda \middle| h^2\right\rangle \end{pmatrix} \succcurlyeq 0$$

and it has to be positive semidefinite. Taking the determinant we obtain $-\langle \Lambda | h\rangle^2 \geq 0$, that implies $\langle \Lambda | h\rangle = 0$. Since this holds for any $h \in \mathbb{R}[\mathbf{x}]$, we have $\Lambda = 0$.

1.3.7 Truncated quadratic modules

We now turn our attention to the main objects of discussion of the thesis: truncated quadratic modules and their duals. Recall the definition of truncated quadratic module at degree $d \in \mathbb{N}$:

$$\mathcal{Q}_d(\mathbf{g}) = \left\{ s_0 + \sum_{j=1}^{r} s_j g_j \in \mathbb{R}[\mathbf{X}]_d \mid s_i \in \Sigma^2 \ \forall i \in \{0, \dots 1\}, \deg s_0 \leq d, \deg s_j g_j \leq d \ \forall j \in \{1, \dots r\} \right\} \quad (1.4)$$

These are convex cones in the space of polynomials of degree $\leq d$.

Unlike the case of SoS, determining whether $f \in \mathcal{Q}(\mathbf{g})$ or $f \notin \mathcal{Q}(\mathbf{g})$ cannot be done solving only one semidefinite program, since the degree of the representation of $f \in \mathcal{Q}(\mathbf{g})$ does not depend only on $\mathbf{g}$ and $\deg f$ (unless $\mathcal{Q}(\mathbf{g})$ is stable). But determining whether $f \in \mathcal{Q}_d(\mathbf{g})$ or $f \notin \mathcal{Q}_d(\mathbf{g})$ is possible using semidefinite programming, as in the case of SoS polynomials. To show this we need to fix some notations: let

- $f = \sum_m f_m b_{2d,m} \in \mathbb{R}[\mathbf{x}]_{2d}$ (f is not necessarily of degree $2d$);
- $g_0 = 1$ and $g_i = \sum_j g_j b_{d_i,j} \in \mathbb{R}[\mathbf{x}]_{d_i}$, where $d_i = \deg g_i$, for $i = 0, \dots r$;
- $N_i = \lfloor \frac{d-d_i}{2} \rfloor$ and $\mathbf{b}_{N_i}$ a basis of $\mathbb{R}[\mathbf{x}]_{N_i}$;

- $X^{(i)} = (X^{(i)}_{j,k})$ an $|\mathbf{b}_{N_i}| \times |\mathbf{b}_{N_i}|$ symmetric matrix.

Then, proceeding as in the SoS case, one can check that $f \in \mathcal{Q}_{2d}(\mathbf{g})$ if and only if the semidefinite program:

$$\begin{cases} \sum_{i=0}^{r} \sum_{\substack{j,k,\ell: \\ b_{N_i,j} b_{N_i,k} b_{d_i,\ell} = b_{2d,m}}} X^{(i)}_{j,k} g_\ell = f_m \quad \forall b_{2d,m} \in \mathbf{b}_{2d} \\ X^{(0)} \succcurlyeq 0, X^{(1)} \succcurlyeq 0, \ldots, X^{(r)} \succcurlyeq 0 \end{cases}$$

is feasible. See for instance [Las15, ch. 6] and [Mar08, ch. 10] for more details, especially in connection with Lasserre's SoS hierarchy (described in Section 1.6).

Proceeding as in the case of SoS, we can restate the problem above using the trace inner product. We see that $f \in \mathcal{Q}_{2d}(\mathbf{g})$ if and only if the semidefinite program:

$$\begin{cases} \sum_{i=0}^{r} \langle X^{(i)}, C^{(i)}_{N_i,m} \rangle = f_m \qquad \forall b_{2d,m} \in \mathbf{b}_{2d} \\ X^{(0)} \succcurlyeq 0, X^{(1)} \succcurlyeq 0, \ldots, X^{(r)} \succcurlyeq 0 \end{cases}$$

is feasible, where $C^{(i)}_{N_i,m}$ is the matrix with (j,k)-entry equal to the coefficient of $b_{2d,\ell}$ in the (j,k)-entry of $g_i C_{N_i}$ (where C_{N_i} is definied as in the SoS case, and $g_i C_{N_i} = \sum_{\ell:\, b_{2d,\ell} \in \mathbf{b}_{2d}} C_{N_i,\ell}$).

1.3.8 Truncated positive linear functionals

We now introduce the dual cones to truncated quadratic modules.

Definition 1.3.8. The cone of *truncated positive linear functionals* is the convex cone $\mathcal{L}_d(\mathbf{d})$ that is dual to $\mathcal{Q}_d(\mathbf{g})$, namely:

$$\mathcal{L}_d(\mathbf{g}) := \mathcal{Q}_d(\mathbf{g})^\vee = \{\Lambda \in (\mathbb{R}[\mathbf{x}]_d)^* \mid \langle \Lambda | q \rangle \geq 0 \text{ for all } q \in \mathcal{Q}_d(\mathbf{g})\}$$

where $\mathcal{Q}_d(\mathbf{g})$ is the truncated quadratic module defined in (1.4). From Conic Duality (Corollary 1.2.3) we deduce that $\mathcal{L}_d(\mathbf{g}) = \overline{\mathcal{L}_d(\mathbf{g})}$ is a closed convex cone, and furthermore $\mathcal{L}_d(\mathbf{g})^\vee = \overline{\mathcal{Q}_d(\mathbf{g})}$.

We can give a semidefinite description of $\mathcal{L}_d(\mathbf{g})$, as we did for $\mathcal{L}(\mathbf{g})$ in the infinite dimensional case. The only difference is that we need to bound the degrees of the polynomials, and thus bound the size of the Hankel matrices. We use the same notations introduced for truncated quadratic modules:

- $d_i = \deg g_i$, for $i = 1, \ldots r$;
- $N_i = \lfloor \frac{d - d_i}{2} \rfloor$.

Then:

$$\mathcal{L}_d(\mathbf{g}) = \{\Lambda \in (\mathbb{R}[\mathbf{x}]_d)^* \mid H_\Lambda^{\lfloor \frac{d}{2} \rfloor} \succcurlyeq 0, H^{N_1}_{g_1 \star \Lambda} \succcurlyeq 0, \ldots, H^{N_r}_{g_r \star \Lambda} \succcurlyeq 0\},$$

and $\mathcal{L}_d(\mathbf{g})$ is a spectrahedron.

Unlike the infinite dimensional case, it is not true that for $\Lambda \in \mathcal{L}_d(\mathbf{g})$, $\langle \Lambda | 1 \rangle = 0 \Rightarrow \Lambda = 0$. For instance, consider $\Lambda \in \mathcal{L}_{2d}(1) = (\Sigma \mathbb{R}[x]^2 \cap \mathbb{R}[x]_{2d})^\vee$, the dual cone of univariate SoS polynomials of degree $\leq 2d$. Then define $\Lambda \in \mathbb{R}[x]_{2d}^*$ as $\langle \Lambda | f \rangle = \left\langle \Lambda \middle| \sum_{i=0}^{2d} f_i x^i \right\rangle = f_{2d}$. Clearly $\Lambda \in \mathcal{L}_{2d}(1)$ and $\langle \Lambda | 1 \rangle = 0$, but $\Lambda \neq 0$. Since we are particularly interested in linear functionals such that $\langle \Lambda | 1 \rangle = 1$ (see Section 1.6), we define:

$$\mathcal{L}_d^{(1)}(\mathbf{g}) := \left\{ \Lambda \in \mathcal{L}_d(\mathbf{g}) \mid \langle \Lambda | 1 \rangle = 1 \right\}.$$

The pathological behavior above shows in particular that $\mathcal{L}_d^{(1)}(\mathbf{g})$, that is an affine section of $\mathcal{L}_d(\mathbf{g})$, is not always a *generating* section, i.e. $\operatorname{cone}(\mathcal{L}_d^{(1)}(\mathbf{g})) \subsetneq \mathcal{L}_{2d}(\mathbf{g})$. But the problem arise only for high degree pseudo-moments, as the following lemma shows.

Lemma 1.3.9. *Let $\Lambda \in \mathbb{R}[\mathbf{x}]_d^*$ be such that $H_\Lambda^{\lfloor \frac{d}{2} \rfloor} \succcurlyeq 0$ and $\langle \Lambda | 1 \rangle = 0$. Then for all $h \in \mathbb{R}[\mathbf{x}]$ such that $\deg h \leq \lfloor \frac{d}{2} \rfloor$, we have $\langle \Lambda | h \rangle = 0$ and thus $\Lambda^{[\lfloor \frac{d}{2} \rfloor]} = 0$. Furthermore, if we denote*

$$\mathcal{L}_d(\mathbf{g})^{[k]} = \{ \Lambda^{[k]} \mid \Lambda \in \mathcal{L}_d(\mathbf{g}) \}$$

where $\Lambda^{[k]}$ is the restriction of Λ to $\mathbb{R}[\mathbf{x}]_k$ (see (1.1)), then $\mathcal{L}_d(\mathbf{g})^{[k]} = \operatorname{cone}(\mathcal{L}_d^{(1)}(\mathbf{g})^{[k]})$ for all d, $\mathbf{g}$ and $k \leq \lfloor \frac{d}{2} \rfloor$.

Proof. We proceed as in the infinite dimensional case. Let Λ and h be as in the hypothesis. Since $H_\Lambda^{\lfloor \frac{d}{2} \rfloor} \succcurlyeq 0$, then choosing a basis of $\mathbb{R}[\mathbf{x}]$ containing 1 and h, the minor of $H_\Lambda^{\lfloor \frac{d}{2} \rfloor}$ indexed by 1 and h has form:

$$\begin{pmatrix} \langle \Lambda | 1 \rangle & \langle \Lambda | h \rangle \\ \langle \Lambda | h \rangle & \left\langle \Lambda \middle| h^2 \right\rangle \end{pmatrix} = \begin{pmatrix} 0 & \langle \Lambda | h \rangle \\ \langle \Lambda | h \rangle & \left\langle \Lambda \middle| h^2 \right\rangle \end{pmatrix} \succcurlyeq 0$$

and it has to be positive semidefinite (notice that it is possible to consider this minor because $\deg h \leq \lfloor \frac{d}{2} \rfloor$). Taking the determinant we obtain $-\langle \Lambda | h \rangle^2 \geq 0$, that implies $\langle \Lambda | h \rangle = 0$. Since this holds for any $h \in \mathbb{R}[\mathbf{x}]$, we have $\Lambda^{[\lfloor \frac{d}{2} \rfloor]} = 0$.

For the second part, consider $\Lambda \in \mathcal{L}_d(\mathbf{g})$. Then $H_\Lambda^{\lfloor \frac{d}{2} \rfloor} \succcurlyeq 0$, and for $k \leq \lfloor \frac{d}{2} \rfloor$, $\langle \Lambda | 1 \rangle = 0$ implies $\Lambda^{[k]} = 0$. Then, if $0 \neq \Lambda^{[k]} \in \mathcal{L}_d(\mathbf{g})^{[k]}$, we have $\langle \Lambda | 1 \rangle = \left\langle \Lambda^{[k]} \middle| 1 \right\rangle > 0$ and $\Lambda^{[k]} = \left\langle \Lambda^{[k]} \middle| 1 \right\rangle \frac{\Lambda^{[k]}}{\langle \Lambda^{[k]} | 1 \rangle}$. Since $\frac{\Lambda^{[k]}}{\langle \Lambda^{[k]} | 1 \rangle} \in \mathcal{L}_d^{(1)}(\mathbf{g})^{[k]}$ and $\left\langle \Lambda^{[k]} \middle| 1 \right\rangle > 0$, we have shown $\mathcal{L}_d(\mathbf{g})^{[k]} = \operatorname{cone}(\mathcal{L}_d^{(1)}(\mathbf{g})^{[k]})$. □

Finally, let us recall other properties of truncated positive linear functionals (see for instance [CF96] or [LLR08]) that will be frequently using. We also prove these statements using our formalism.

Lemma 1.3.10. *Let $\Lambda \in \mathbb{R}[\mathbf{x}]_{2d}^*$ be such that $H_\Lambda^2 \succcurlyeq 0$. Then:*

(i) for $h \in \mathbb{R}[\mathbf{x}]$ of degree $\leq d$, $\left\langle \Lambda \middle| h^2 \right\rangle = 0$ implies $h \in \operatorname{Ann}_d(\Lambda)$.

(ii) if $f \in \operatorname{Ann}_d(\Lambda)$ and $\deg gf \leq d-1$, then $gf \in \operatorname{Ann}_d(\Lambda)$.

Proof. For the first point, let $h \in \mathbb{R}[\mathbf{x}]$ of degree ≤ 2, be such that $\left\langle \Lambda \middle| h^2 \right\rangle = 0$. We want to prove that $h \in \operatorname{Ann}_d(\Lambda)$, or more explicitly that $\left\langle (f \star \Lambda)^{[d]} \middle| p \right\rangle = \langle \Lambda | f p \rangle = 0$ for all $p \in \mathbb{R}[\mathbf{x}]_d$. Recall also that $H_\Lambda^2 \succcurlyeq 0$ if and only if $\left\langle \Lambda \middle| p^2 \right\rangle \geq 0$ for all $p \in \mathbb{R}[\mathbf{x}]_d$.

Now, for all $t \in \mathbb{R}$ we have:

$$0 \leq \left\langle \Lambda \middle| (h+tp)^2 \right\rangle = 2t \langle \Lambda | hp \rangle + t^2 \left\langle \Lambda \middle| p^2 \right\rangle.$$

Then the univariate polynomial $t \mapsto 2t\langle \Lambda | hp\rangle + t^2 \left\langle \Lambda \middle| p^2 \right\rangle$ has a double root at $t = 0$, and this implies $\langle \Lambda | hp \rangle = 0$ for all $p \in \mathbb{R}[\mathbf{x}]_d$. Therefore $h \in \mathrm{Ann}_d(\Lambda)$.

For the second point, by induction on the degree of g it is enough to prove that, if $f \in \mathrm{Ann}_d(\Lambda)$ and $\deg x_i f \leq d-1$, then $x_i f \in \mathrm{Ann}_d(\Lambda)$. From the first point, to prove this it is enough to show that $\left\langle \Lambda \middle| (x_i f)^2 \right\rangle = 0$. But this is clear, since $\deg x_i^2 f \leq t$ and:

$$\left\langle \Lambda \middle| (x_i f)^2 \right\rangle = \left\langle (f \star \Lambda)^{[t]} \middle| x_i^2 f \right\rangle = \left\langle H_\Lambda^d(f) \middle| x_i^2 f \right\rangle = \left\langle 0 \middle| x_i^2 f \right\rangle = 0,$$

which concludes the proof. □

Point (ii) in the previous lemma says that the truncated annihilator $\mathrm{Ann}_d(\Lambda)$ enjoys an ideal-like property. This is coherent with the infinite dimansional case, since the annihilator $\mathrm{Ann}(\Lambda)$ is an ideal.

1.4 The moment problem

We briefly recall the *Moment Problem*, and refer to [Sch17] for additional references and more details.

Classically, the Moment Problem on the real line can be stated as follows. Given a sequence of real numbers $\Lambda = (\Lambda_n)_{n\in\mathbb{N}}$, there exists a (positive) Borel measure $\mu \in \mathcal{M}(\mathbb{R})$ such that $\Lambda_n = \int x^n \,\mathrm{d}\mu$ for all $n \in \mathbb{N}$? The multivariate extension is straightforward: given a sequence of real numbers $(\Lambda_\alpha)_{\alpha\in\mathbb{N}^n}$, there exists a Borel measure $\mu \in \mathcal{M}(\mathbb{R}^n)$ such that $\Lambda_n = \int \mathbf{x}^\alpha \,\mathrm{d}\mu = \int x_1^{\alpha_1} \dots x_n^{\alpha_n} \,\mathrm{d}\mu$ for all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$? As we have seen in Section 1.2.2, the numbers $\int \mathbf{x}^\alpha \,\mathrm{d}\mu$ are called *moments* of μ, and thus we could restate the Moment Problem as follows: is the sequence $(\Lambda_\alpha)_{\alpha\in\mathbb{N}^n}$ a sequence of moments?

A natural additional condition that we can require is a prescribed support for the representing measure. This is the so called *Strong Moment Problem*: given a sequence of real numbers $(\Lambda_\alpha)_{\alpha\in\mathbb{N}^n}$ and $D \subset \mathbb{R}^n$, there exists a Borel measure $\mu \in \mathcal{M}(D)$ such that $\Lambda_n = \int \mathbf{x}^\alpha \,\mathrm{d}\mu$ for all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$?

We have described in Section 1.2.2 the isomorphism between pseudo-moments sequences and linear funcitonals acting on the polynomial ring, and recall that we denote Λ_μ the linear functionals induced by measures μ. Therefore, it is convenient to restate the (Strong) Moment Problem as follows: given $\Lambda \in \mathbb{R}[\mathbf{x}]^*$, there exists μ (with prescribed support) such that $\Lambda = \Lambda_\mu$? Or in more concise words, is Λ a moment linear funcitonal?

It is then interesting to study sufficient and necessary conditions for Λ to be equal to Λ_μ, for some $\mu \in \mathcal{M}(D)$. A key result in this direction is Haviland's Theorem, Theorem 1.3.1: it states that, if D is compact, Λ is a moment linear functional with representing measure supported on D if and only if $\langle \Lambda | f \rangle \geq 0$ for all $f \in \mathrm{Pos}(D)$.

The drawback of this theorem is that checking this condition is a computationally hard task, since the convex cone $\mathrm{Pos}(D)$ has no simple representation. Therefore, we would like to replace $\mathrm{Pos}(D)$ in Haviland's Theorem with a simpler subcone, in such a way checking the condition is simpler.

A motivation to study Sum of Squares representations, and more precisely quadratic modules and preorderings, is that these subcones of $\mathrm{Pos}(\mathcal{S}(\mathbf{g}))$ have the property thar we are looking for. Indeed:

- Schmüdgen's Theorem (Theorem 1.3.6) says that Λ is a moment linear functional, with representing measure supportend on $\mathcal{S}(\mathbf{g})$, if and only if $\langle \Lambda | p \rangle \geq 0$ for all $p \in \mathcal{O}(\mathbf{g})$, when $\mathcal{S}(\mathbf{g})$ is compact;

- Putinar's Theorem (Theorem 1.3.7) says that Λ is a moment linear functional, with representing measure supportend on $\mathcal{S}(\mathbf{g})$, if and only if $\langle \Lambda | q \rangle \geq 0$ for all $q \in \mathcal{Q}(\mathbf{g})$, when $\mathcal{Q}(\mathbf{g})$ is Archimedean.

Another natural question is the question of *uniqueness*, i.e.: given a moment linear functional Λ, is the measure μ inducing $\Lambda = \Lambda_\mu$ unique? If the answer is yes, we say that the Moment Problem is *determinate*. As a consequence of the Stone-Weierstrass approximation theorem, the moment problem is determinate when the measure has compact support. We refer to [Sch17, ch. 14] for a more detailed discussion of determinacy.

1.4.1 The truncated moment problem

Up to now we focused our attention to the *infinite dimensional* case, since our sequences $(\Lambda_\alpha)_{\alpha\in\mathbb{N}^n}$ have infinitely many pseudo-moments, or equivalently because the linear functionals are defined on the full polynomial ring $\mathbb{R}[\mathbf{x}]$. However, from the practical point of view it is necessary to restrict our attention to the *finite dimensional* case, i.e. when we have only finitely many coefficients available, or equivalently, when the linear functional Λ is defined only on a finite dimensional vector subspace of $\mathbb{R}[\mathbf{x}]$. In particular, we will restict our attention to the subspace of polynomial of bounded degree, or in other words to truncates sequences indexed by multi-indices of bounded sum.

Then, the *Truncated* Moment Problem can be stated as follows: given a truncated sequence $\Lambda = (\Lambda_\alpha)_{|\alpha|\leq d}$, there exists a (positive) Borel measure $\mu \in \mathcal{M}(\mathbb{R})$ such that $\Lambda_\alpha = \int \mathbf{x}^\alpha \, \mathrm{d}\mu$ for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq d$? Or equivalantly, given $\Lambda \in \mathbb{R}[\mathbf{x}]_d^*$, there exists μ such that $\Lambda = \Lambda_\mu^{[d]}$? As in the infinite dimensional case, we speak of *Strong Truncated Moment Problem* when we investigate representation of linear functionals on $\mathbb{R}[\mathbf{x}]_d$ as moment linear functionals with a prescribed support.

In the truncated setting, Dirac measures δ_x (or equivalently evaluations $\mathbf{e}_x$) play a special role, as shown by the Richter-Tchakaloff theorem.

Theorem 1.4.1 (Richter-Tchakaloff)**.** *Let $\Lambda = \Lambda_\mu^{[d]} \in \mathbb{R}[\mathbf{x}]_d^*$, $\mu \in \mathcal{M}(D)$, be a truncated moment linear functional. Then Λ is represented by an atomic measure with atoms in D: there exists $\xi_1, \dots \xi_m \in D$ and $a_1, \dots, a_m \in \mathbb{R}_{>0}$ such that $\Lambda = a_1 \mathbf{e}_{\xi_1} + \cdots + a_m \mathbf{e}_{\xi_m}$. Moreover, $\mathcal{M}(D)^{[d]} =$ $\mathrm{cone}(\mathbf{e}_\xi : \xi \in D)^{[d]}$.*

We refer to [Sch17, th. 1.24] for the proof, and to [DS22] for an historical discussion. Theorem 1.4.1 show the existence of the so-called *quadrature rules* or *quadrature formulas*, that are used to compute integrals of polynomials of bounded degree.

Unfortunately, Richter-Tchakaloff theorem does not give a test to determine whether a truncated linear funcitonal is a moment linear functional or not. The most important test in this direction is the celebrated Curto-Fialkow Flat Extension Theorem.

Theorem 1.4.2 (Flat Extension Theorem, [CF96]). *Let $\Lambda \in \mathbb{R}[\mathbf{x}]_{2d}^*$ be a truncated linear functional. Then Λ is induced by an atomic measure with* $\operatorname{rank} H_\Lambda^{2d}$ *atoms if and only if $H_\Lambda^d \succcurlyeq 0$ and there exists $\widetilde{\Lambda} \in \mathbb{R}[\mathbf{x}]_{2d+2}^*$ such that:*

- $\widetilde{\Lambda}^{[2d]} = \Lambda$*;*
- $\operatorname{rank} H_\Lambda^d = \operatorname{rank} H_{\widetilde{\Lambda}}^{d+1}$.

We refer to [Lau09] and reference therein for a survey on flat extension properties (and especially their use in Polynomial Optimization), and to [LM09; Mou18] for more general versions of this theorem.

1.5 Conic programming

We now briefly recall the duality theory for linear programming over cones, a general framework for polynomial optimization problems, following [Bar02]. In the follwing we will show how semidefinite programming, Polynomial Optimization problems and Lasserre's hierarchies can be described using this framework.

A *duality* or *duality pairing* of vector spaces is a non degenerate bilinear map $\langle \cdot, \cdot \rangle \colon E \times F \to \mathbb{R}$, where E and F are real vector spaces. For instance, take:

- E any vector space and $F = E^*$ be the linear functionals on E, with duality pairing $\langle e, f \rangle = f(e)$ given by the natural application of $f \in F$ to $e \in E$ (one can replace E^* with E' when E is a topological vector space);
- $E = F = \mathbb{R}^n$ with the Eucliden inner product: $\langle x, y \rangle = \sum_i x_i y_i$;
- $E = F = \mathcal{S}^n$ the space of symmetric matrices with the trace inner product: $\langle X, Y \rangle = \operatorname{tr}(XY)$.

Let $\langle \cdot, \cdot \rangle_1 \colon E_1 \times F_1 \to \mathbb{R}$ and $\langle \cdot, \cdot \rangle_2 \colon E_2 \times F_2 \to \mathbb{R}$ be dualities of vector spaces, and let $K_1 \subset E_1$, $K_2 \subset E_2$ be convex cones. Denote $K_i^* = \{ f \in F_i \mid \langle k, f \rangle_i \geq 0 \ \forall k \in \mathbb{K}_i \}$ the dual convex cone with respect to $\langle \cdot, \cdot \rangle_i$ (as a subset of F_i). Let $A \colon E_1 \to E_2$ be linear and denote $A^* \colon F_2 \to E_2$ the *adjoint* of A, that is the linear map such that $\langle Ae, f \rangle_2 = \langle e, A^* f \rangle_1$ for all $e \in E_1$ and $f \in F_2$.

We fix $c \in F_1$ and $b \in E_2$. With the notation above, we define a *pair of linear programming problems*.

$$
\begin{aligned}
\textit{Primal Problem:} \quad & \text{Find:} && \gamma = \inf_{x \in E_1} \langle x, c \rangle_1 \\
& \text{subject to:} && Ax - b \in K_2 \\
& && x \in K_1
\end{aligned}
$$

$$
\begin{aligned}
\textit{Dual Problem:} \quad & \text{Find:} && \beta = \sup_{\ell \in F_2} \langle b, l \rangle_2 \\
& \text{subject to:} && c - A^* \ell \in K_1^* \\
& && \ell \in K_2^*
\end{aligned}
$$

If $x \in E_1$ satisfies $Ax - b \in K_2$ and $x \in K_1$, then x is called *primal feasible* (or simply *feasible*, when clear from the context). Similarly, if $\ell \in F_2$ satisfies $c - A^*\ell \in K_1^*$ and $\ell \in K_2^*$, then ℓ is called *dual fesible* (or simply *feasible*, when clear from the context).

We recall the main general results relating γ and β.

Theorem 1.5.1 ([Bar02, th. 6.2]). *With the notations above:*

(i) (Weak Duality) For any primal feasible $x \in E_1$ and dual feasible $\ell \in F_2$, we have $\langle x, c\rangle_1 \leq \langle b, \ell\rangle_2$ and $\gamma \geq \beta$. Furthermore, if $\langle x, c\rangle_1 = \langle b, \ell\rangle_2$ then x is primal optimal (i.e. $\langle x, c\rangle_1 = \gamma$) and ℓ is dual optimal (i.e. $\langle b, \ell\rangle_2 = \beta$) and $\gamma = \beta$ (strong duality).

(ii) (Optimality Criterion) If x is primal feasible and ℓ is dual feasible such that $\langle x, c - A^*\ell\rangle_1 = 0$ and $\langle Ax - b, \ell\rangle_2 = 0$, then x is primal optimal, ℓ is dual optimal and $\gamma = \beta$ (strong duality).

(iii) (Complementary Slackness) If x is primal optimal, ℓ is dual optimal and $\gamma = \beta$, then $\langle x, c - A^*\ell\rangle_1 = 0$ and $\langle Ax - b, \ell\rangle_2 = 0$.

We now desribe semidefinite programming using this framework.

1.5.1 Semidefinite programming

Let $E_1 = F_1 = \mathbb{R}^m$ (m will the codimension of the affine section of $\mathcal{S}_+^n$ we are going to consider), let $\langle x, y\rangle_1 = \sum_i x_i y_i$ the Euclidean inner product. Let $K_1 = \mathbb{R}^m$ and $K_1^* = \{0\}$. Let $E_2 = F_2 = \mathcal{S}^n$ the space of symmetric matrices with the trace inner product: $\langle A, B\rangle_2 = \mathrm{tr}(AB)$, and finally $K_2 = K_2^* = \mathcal{S}_+^n$.

A linear transformation $A\colon \mathbb{R}^m \to \mathcal{S}^n$ can be written as $Ax = x_1 A_1 + \cdots + x_m A_m$, for some $A_i \in \mathcal{S}^n$, and the adjoint has form $A^*X = (\mathrm{tr}(XA_1), \ldots, \mathrm{tr}(XA_m))$. Then a Semidefinite Program has the following description (in canonical forms), given $\lambda \in \mathbb{R}^m$ and $C, A_1, \ldots A_m \in \mathcal{S}^n$:

Primal: Find: $\displaystyle \beta = \inf_{\ell \in \mathbb{R}^m} \sum_i \lambda_i \ell_i$

subject to: $\displaystyle \sum_i \ell_i A_i - C \in \mathcal{S}_+^n$, $\ell \in \mathbb{R}^m$

Dual: Find: $\displaystyle \gamma = \sup_{X \in \mathcal{S}^n} \mathrm{tr}(CX)$

subject to: $\lambda_i - \mathrm{tr}(A_i X) = 0\ \forall i$, $X \in \mathcal{S}_+^n$

It is clear that primal and dual feasible points are spectrahedra.

1.6 Polynomial optimization

Let $f, g_1, \ldots, g_s \in \mathbb{R}[\mathbf{x}]$ be polynomials in the indeterminates $x_1, \ldots, x_n$ with real coefficients. The goal of Polynomial Optimization is to find:

$$f^* := \inf \left\{ f(x) \in \mathbb{R} \mid x \in \mathbb{R}^n,\ g_i(x) \geq 0 \text{ for } i = 1, \ldots, s \right\}, \tag{1.5}$$

that is the infimum f^* of the *objective function* f on the *basic closed semialgebraic set* $S = \mathcal{S}(\mathbf{g}) = \{x \in \mathbb{R}^n \mid g_i(x) \geq 0 \text{ for } i = 1, \ldots, s\}$. It is a general problem, which appears in many contexts (e.g. real solution of polynomial equations, ...) and with many applications. To cite a few of them: in graph theory [LV21], network optimization design [MH15], control [HK14], ... See [Las10] for a more comprehensive list.

To solve this NP-hard problem, Lasserre [Las01] proposed to use two hierarchies of finite dimensional convex cones depending on an order $d \in \mathbb{N}$ and he proved, for Archimedean quadratic modules, the convergence when $d \to \infty$ of the optima associated to these hierarchies to the minimum f^* of f on S. We will describe these hierarchies in the following sections.

This approach has many interesting properties (see e.g. [Las15], [Lau09], [Mar08]). It was proposed with the aim to recover the infimum f^* and, if this infimum is reached, the set of minimizers $\{\xi \in S \mid f(\xi) = f^*\}$. The extraction of minimizers is strongly connected to the so called *flat truncation* property, see Section 1.4.

1.6.1 Inner approximations of positive polynomials

We want to find lower approximations of f^*. This is naturally possible, thanks to the following observation:

$$\begin{aligned} f^* &= \inf\left\{ f(x) \in \mathbb{R} \mid x \in \mathcal{S}(\mathbf{g}) \right\} \\ &= \sup\left\{ \lambda \in \mathbb{R} \mid f - \lambda \in \operatorname{Pos}(\mathcal{S}(\mathbf{g})) \right\}. \end{aligned}$$

Therefore, replacing the cone of positive polynomials $\operatorname{Pos}(\mathcal{S}(\mathbf{g}))$ with any subcone we will define a lower approximation of f^*.

Lasserre's sum of squares hierarchy

We choose as subcone of $\operatorname{Pos}(\mathcal{S}(\mathbf{g}))$ the truncated quadratic module. Indeed, for any $d \in \mathbb{N}$, $\mathcal{Q}_{2d}(\mathbf{g}) \subset \operatorname{Pos}(\mathcal{S}(\mathbf{g}))$.

We define the *Lasserre's SoS relaxation of order d* of problem (1.5) as $\mathcal{Q}_{2d}(\mathbf{g})$ and the supremum:

$$f^*_{\mathrm{SoS},d} := \sup\left\{ \lambda \in \mathbb{R} \mid f - \lambda \in \mathcal{Q}_{2d}(\mathbf{g}) \right\} \leq f^*. \tag{1.6}$$

When necessary we will replace $\mathbf{g}$ by $\Pi\mathbf{g}$ (that is $\mathcal{Q}(\mathbf{g})$ by $\mathcal{O}(\mathbf{g})$).

Notice that this construction works for any $d \in \mathbb{N}$. Thus, we have defined a *hierarchy* of inner approximations of $\operatorname{Pos}(\mathcal{S}(\mathbf{g}))$:

$$\mathcal{Q}_{2d}(\mathbf{g}) \subset \mathcal{Q}_{2d+2}(\mathbf{g}) \subset \cdots \subset \operatorname{Pos}(\mathcal{S}(\mathbf{g}))$$

and therefore we have a sequence of lower approximations of f^*:

$$f^*_{\mathrm{SoS},d} \leq f^*_{\mathrm{SoS},d+1} \leq \cdots \leq f^*.$$

The convergence of the lower approximations to f^* depends on the possibility to approximate positive polynomials using the hierarchy of subcones of $\operatorname{Pos}(\mathcal{S}(\mathbf{g}))$. Concretely, for the case of the Lasserre's SoS relaxation, if the quadratic module $\mathcal{Q}(\mathbf{g})$ is Archimedean then $\lim_{d\to\infty} f^*_{\mathrm{SoS},d} = f^*$. Indeed, for any $\varepsilon > 0$, $f - f^* + \varepsilon > 0$ on $\mathcal{S}(\mathbf{g})$. Therefore, from Putinar's Positivstellensatz (Theorem 1.1.22) there exists $d = d(\varepsilon)$ such that $f - f^* + \varepsilon \in \mathcal{Q}_{2d}(\mathbf{g})$. This shows that $f^* - \varepsilon \leq f^*_{\mathrm{SoS},d}$, and we have convergence of the lower approximations as $d \to \infty$.

1.6.2 Outer approximations of measures

We now turn our attention to a different description of f^*:

$$\begin{aligned} f^* &= \inf\Big\{ f(x) \in \mathbb{R} \mid x \in \mathcal{S}(\mathbf{g}) \Big\} \\ &= \inf\Big\{ \int f \,\mathrm{d}\mu \in \mathbb{R} \mid \mu \in \mathcal{M}(\mathcal{S}(\mathbf{g})),\ \int \mathrm{d}\mu = 1 \Big\} \end{aligned}$$

Therefore, replacing the cone of measures with any cone containing $\mathcal{M}(\mathcal{S}(\mathbf{g}))$ will define a lower approximation of f^*. To define this cone, recall that $\mathcal{M}(\mathcal{S}(\mathbf{g})) = \mathrm{Pos}(\mathcal{S}(\mathbf{g}))^\vee$. Therefore, taking the dual of any inner approximation of $\mathrm{Pos}(\mathcal{S}(\mathbf{g}))$ will define a lower approximation of f^*.

Lasserre's moment hierarchy

We choose to take the dual of the cones used in Lasserre's SoS hierarchy. Concretely, we choose:

$$\mathcal{L}_{2d}(\mathbf{g}) = \mathcal{Q}_{2d}(\mathbf{g})^\vee = \{ \Lambda \in \mathbb{R}[\mathbf{x}]^* \mid \langle \Lambda | q \rangle \geq 0 \text{ for all } q \in \mathcal{Q}_{2d}(\mathbf{g}) \} \supset \mathcal{M}(\mathcal{S}(\mathbf{g}))^{[2d]},$$

and in particular we consider an affine hyperplane section of $\mathcal{L}_{2d}(\mathbf{g})$:

$$\mathcal{L}_{2d}^{(1)}(\mathbf{g}) = \Big\{ \Lambda \in \mathcal{L}_{2d}(\mathbf{g}) \mid \langle \Lambda | 1 \rangle = 1 \Big\}.$$

We define the *moment relaxation of order d* of problem (1.5) as $\mathcal{L}_{2d}(\mathbf{g})$ and the infimum:

$$f^*_{\mathrm{Mom},d} := \inf\Big\{ \langle \Lambda | f \rangle \in \mathbb{R} \mid \Lambda \in \mathcal{L}_{2d}^{(1)}(\mathbf{g}) \Big\}. \tag{1.7}$$

Notice that this construction works for any $d \in \mathbb{N}$. Thus, we have defined a *hierarchy* of outer approximations of $\mathcal{M}(\mathcal{S}(\mathbf{g}))^{[d]}$:

$$\mathcal{L}_{2d}(\mathbf{g}) \supset \mathcal{L}_{2d+2}(\mathbf{g})^{[2d]} \supset \ldots \mathcal{M}(\mathcal{S}(\mathbf{g}))^{[2d]}.$$

Notice also that we have to take the restriction to be able to compare linear functionals acting on different spaces, or equivalently to compare pseudo-moment sequences of different lengths. Since $\langle \Lambda | f \rangle = \left\langle \Lambda^{[2d]} \middle| f \right\rangle$ for any d such that $2d \geq \deg f$, we have therefore a sequence of lower approximations of f^*:

$$f^*_{\mathrm{Mom},d} \leq f^*_{\mathrm{Mom},d+1} \leq \cdots \leq f^*.$$

We verify that $f^*_{\mathrm{SoS},d} \leq f^*_{\mathrm{Mom},d}$ for every d. Indeed, if $f - \lambda \in \mathcal{Q}_{2d}(\mathbf{g})$ then $0 \leq \langle \Lambda | f - \lambda \rangle = \langle \Lambda | f \rangle - \lambda$ for all $\Lambda \in \mathcal{L}_{2d}^{(1)}(\mathbf{g})$, and thus $\lambda \leq \langle \Lambda | f \rangle$ for all Λ. This implies $f^*_{\mathrm{SoS},d} \leq f^*_{\mathrm{Mom},d}$.

From the previous result, we can deduce convergence of $f^*_{\mathrm{Mom},d}$ to f^* from the convergence of $f^*_{\mathrm{SoS},d}$ to f^*. In particular, if $\mathcal{Q}(\mathbf{g})$ is Archimedean then $\lim_{d\to\infty} f^*_{\mathrm{Mom},d} = f^*$.

1.6.3 Polynomial optimization as conic programming

Recall the Polynomial Optimization problem, given $f, g_1, \ldots, g_r$, find:

$$\begin{aligned} f^* &= \inf\Big\{ f(x) \in \mathbb{R} \mid x \in \mathcal{S}(\mathbf{g}) \Big\} \\ &= \sup\Big\{ \lambda \in \mathbb{R} \mid f - \lambda \in \operatorname{Pos}(\mathcal{S}(\mathbf{g})) \Big\} \\ &= \inf\Big\{ \int f \,\mathrm{d}\mu \in \mathbb{R} \mid \mu \in \mathcal{M}(\mathcal{S}(\mathbf{g})), \int \mathrm{d}\mu = 1 \Big\} \end{aligned}$$

Let $E_1 = \mathbb{R}[\mathbf{x}]^*$, $F_1 = \mathbb{R}_g$ and $\langle \cdot,\cdot \rangle_1 = \langle \cdot|\cdot \rangle$ be the application of the linear functional to its argument $K_1 = \mathcal{M}(\mathcal{S}(\mathbf{g}))$, $K_1^* = \operatorname{Pos}(\mathcal{S}(\mathbf{g}))$, $E_2 = \mathbb{R}$, $F_2 = \mathbb{R}$ and $\langle \cdot,\cdot \rangle_2$ be the Euclidean inner product, and finally $K_2 = \{0\}$ and $K_2^* = \mathbb{R}$. Let $A\colon \mathbb{R}[\mathbf{x}]^* \to \mathbb{R}$, $\Lambda \mapsto \langle \Lambda|1 \rangle$, and then $A^*\colon \mathbb{R} \to \mathbb{R}[\mathbf{x}]$, $\lambda \mapsto \lambda$ (as a constant polynomial) is the adjoint. Let $c = f \in \mathbb{R}[g]$ and $b = 1 \in \mathbb{R}[\mathbf{x}]$. Then we can rewrite the polynomial optimization problem as:

Primal: Find: $f^* = \inf_{\Lambda \in \mathbb{R}[\mathbf{x}]^*} \langle \Lambda|f \rangle$
subject to: $\langle \Lambda|1 \rangle - 1 = 0$
$\Lambda = \Lambda_\mu \in \mathcal{M}(\mathcal{S}(\mathbf{g}))$

Dual: Find: $f^* = \sup_{\lambda \in \mathbb{R}} \lambda$
subject to: $f - \lambda \in \operatorname{Pos}(\mathcal{S}(\mathbf{g}))$
$\lambda \in \mathbb{R}$

Notice that it is possible to switch the role of the primal and the dual, proceeding similarly and using the fact that $\sup A = -\inf -A$ for all $A \subset \mathbb{R}$:

Primal: Find: $-f^* = \inf_{\lambda \in \mathbb{R}} -\lambda$
subject to: $f - \lambda \in \operatorname{Pos}(\mathcal{S}(\mathbf{g}))$
$\lambda \in \mathbb{R}$

Dual: Find: $-f^* = \sup_{\Lambda \in \mathbb{R}[\mathbf{x}]^*} -\langle \Lambda|f \rangle$
subject to: $1 - \langle \Lambda|1 \rangle = 0$
$\Lambda = \Lambda_\mu \in \mathcal{M}(\mathcal{S}(\mathbf{g}))$

Above, we have chosen to present the problem using only polynomials and seeing measures as linear functionals acting on polynomials. But we could have replaced positive polynomials with positive continuous functions, and then we would have seen the Borel measures as a cone in the space of Borel *signed* measures. See [Las15, ch. 5] for more details.

1.6.4 Lasserre's relaxations as conic programming

We now adapt Section 1.6.3 to Lasserre's hierarchies in polynomial optimization. If we proceed in the same way, we have:

Primal: Find: $\gamma = f^*_{\mathrm{Mom},d} = \inf_{\Lambda \in \mathbb{R}[\mathbf{x}]^*_{2d}} \langle \Lambda|f \rangle$
subject to: $\langle \Lambda|1 \rangle - 1 = 0$
$\Lambda \in \mathcal{L}_{2d}(\mathbf{g})$

Dual: Find: $\beta = \sup_{\lambda \in \mathbb{R}} \lambda$ (1.8)
subject to: $f - \lambda \in \overline{\mathcal{Q}_{2d}(\mathbf{g})}$
$\lambda \in \mathbb{R}$

Above, we have $K_1 = \mathcal{L}_{2d}(\mathbf{g})$ and thus $K_1^\vee = \overline{\mathcal{Q}_{2d}(\mathbf{g})}$. This is not the Lasserre's SoS relaxation, since we are using $\overline{\mathcal{Q}_{2d}(\mathbf{g})}$ and not $\mathcal{Q}_{2d}(\mathbf{g})$ (see Example 1.3.4 for an example where a truncated quadratic module is not closed): this shows that the natural object that one has to consider when studying the moment hierarchy is $\overline{\mathcal{Q}_{2d}(\mathbf{g})}$, and not $\mathcal{Q}_{2d}(\mathbf{g})$. The difference between $\overline{\mathcal{Q}_{2d}(\mathbf{g})}$ and $\mathcal{Q}_{2d}(\mathbf{g})$ will be important in Chapter 3.

The way one usually obtains the SoS relaxation dualizing the moment relaxation is slightly different, using cones of positive semidefinite matrices. We will describe this procedure in the remaining of the section, but first notice that, reversing the role of primal and dual problem as in Section 1.6.3, we obtain the correct hierarchies:

$$\begin{aligned} \textit{Primal:} \quad \text{Find:} \quad & -f^*_{\mathrm{SoS},d} = \inf_{\lambda\in\mathbb{R}} -\lambda \\ \text{subject to:} \quad & f-\lambda \in \mathcal{Q}_{2d}(\mathbf{g}) \\ & \lambda\in\mathbb{R} \end{aligned} \qquad \begin{aligned} \textit{Dual:} \quad \text{Find:} \quad & -f^*_{\mathrm{Mom},d} = -\sup_{\Lambda\in\mathbb{R}[\mathbf{x}]^*_{2d}} \langle \Lambda | f\rangle \\ \text{subject to:} \quad & 1-\langle\Lambda|1\rangle = 0 \\ & \Lambda \in \mathcal{L}_{2d}(\mathbf{g}) \end{aligned}$$

In this sense, it is more natural to start from the SoS hierarchy and then deduce the moment hierarchy, than doing the opposite.

We now describe Lasserre's relaxations using semidefinite programming. We will need to consider the Hankel map A_g, that associate to a linear functional the localizing matrix $H^N_{g\star\Lambda}$:

$$\begin{aligned} A_g \colon \mathbb{R}[\mathbf{x}]^*_{2d} &\to \mathcal{S}^m \subset \hom_{\mathbb{R}}(\mathbb{R}[\mathbf{x}]_N, \mathbb{R}[\mathbf{x}]^*_N) \\ \Lambda &\mapsto A_g(\Lambda) := H^N_{g\star\Lambda} \end{aligned}$$

where $m = \dim \mathbb{R}[\mathbf{x}]_N$. We also need to consider its adjoint A^*_g with respect to the trace inner product. We describe this adjoint in the following.

Let $X = \mathrm{vec}(h)\mathrm{vec}(h)^t \in \mathcal{S}^m$, $h = \mathbf{b}^t_m \mathrm{vec}(h)$ as in (1.2), be a rank one symmetric matrix. Then:

$$\begin{aligned} \mathrm{tr}\Big(A_g(\Lambda)X\Big) &= \mathrm{tr}\Big(H^m_{g\star\Lambda}X\Big) = \mathrm{tr}\Big(H^m_{g\star\Lambda}\,\mathrm{vec}(h)\mathrm{vec}(h)^t\Big) \\ &= \mathrm{tr}\Big(\mathrm{vec}(h)^t H^m_{g\star\Lambda}\,\mathrm{vec}(h)\Big) = \mathrm{vec}(h)^t H^m_{g\star\Lambda}\,\mathrm{vec}(h) \\ &= \Big\langle H^m_{g\star\Lambda}(h)\Big|h\Big\rangle = \Big\langle \Lambda\Big| g h^2\Big\rangle \\ &= \Big\langle \Lambda \Big| g\, \mathbf{b}^t_m \mathrm{vec}(h)\mathrm{vec}(h)^t \mathbf{b}_m\Big\rangle = \Big\langle \Lambda\Big| g\,\mathbf{b}^t_m X \mathbf{b}_m\Big\rangle. \end{aligned}$$

Since any symmetric matrix of rank r can be written as $\sum_{i=1}^r a_i X^{(i)}$, $a_i\in\mathbb{R}$ and $X_i = \mathrm{vec}(h_i)\mathrm{vec}(h_i)^t$, this shows by linearity that the adjoint is:

$$\begin{aligned} A^*_g \colon \mathcal{S}^m &\to \mathbb{R}[\mathbf{x}]_{2d} \\ X &\mapsto A^*_g(X) = g\,\mathbf{b}^t_m X \mathbf{b}_m \end{aligned}$$

Now we can use this adjoint map to describe the SoS and moment hierarchies. Recall that the feasible set of the moment relaxation of order d is:

$$\mathcal{L}^{(1)}_{2d}(\mathbf{g}) = \{\Lambda \in (\mathbb{R}[\mathbf{x}]_d)^* \mid \langle\Lambda|1\rangle = 1,\ H^d_\Lambda \succcurlyeq 0, H^{N_1}_{g_1\star\Lambda} \succcurlyeq 0, \ldots, H^{N_r}_{g_r\star\Lambda} \succcurlyeq 0\}.$$

Then we want to impose the constraint $\langle\Lambda|1\rangle = 1$, and the positive semidefinite constraints. This means to consider the map

$$A(\Lambda) := (\langle\cdot|1\rangle \times A_1 \times A_{g_1} \times \cdots \times A_{g_r})(\Lambda) = \Big(\langle\Lambda|1\rangle, H^d_\Lambda, H^{N_1}_{g_1\star\Lambda}, \ldots, H^{N_r}_{g_r\star\Lambda}\Big),$$

the cone $K_2 = \{0\} \times \mathcal{S}_+^{\dim \mathbb{R}[\mathbf{x}]_d} \times \mathcal{S}_+^{\dim \mathbb{R}[\mathbf{x}]_{N_1}} \times \cdots \times \mathcal{S}_+^{\dim \mathbb{R}[\mathbf{x}]_{N_1}}$, and the natural inner product $\langle \cdot, \cdot \rangle_1$ defined as sum of the natural inner product in every component. Finally, setting $K_1 = \mathbb{R}[\mathbf{x}]_{2d}^*$, $c = f$ and $b = (1, 0, 0, \ldots, 0)$ we obtain:

$$\begin{aligned} &\textit{Primal Problem:} \quad \text{Find:} \quad && f_{\text{Mom},d}^* = \inf \langle \Lambda | f \rangle = \sum_\alpha \Lambda_\alpha f_\alpha && (1.9) \\ &\text{subject to:} && \langle \Lambda | 1 \rangle - 1 = 0 \\ &&& H_\Lambda^d \succcurlyeq 0, H_{g_1 \star \Lambda}^{N_1} \succcurlyeq 0, \ldots, H_{g_r \star \Lambda}^{N_r} \succcurlyeq 0 \\ &&& \Lambda \in \mathbb{R}[\mathbf{x}]_{2d}^* \end{aligned}$$

which is the usual moment relaxation 1.7. To compute the dual, we have to consider the adjoint:

$$A^*(\lambda, X^{(0)}, X^{(1)}, \ldots, X^{(r)}) = \lambda + \sum_{i=0}^r g_i \mathbf{b}_{N_i}^t X^{(i)} \mathbf{b}_{N_i},$$

and we obtain:

$$\begin{aligned} &\textit{Dual of } (1.9)\textit{:} \quad \text{Find:} \quad && f_{\text{SoS},d}^* = \sup \lambda && (1.10) \\ &\text{subject to:} && f - \lambda - \sum_{i=0}^r g_i \mathbf{b}_{N_i}^t X^{(i)} \mathbf{b}_{N_i} = 0 \\ &&& \lambda \in \mathbb{R}, X^{(0)} \succcurlyeq 0, X^{(1)} \succcurlyeq 0, \ldots, X^{(r)} \succcurlyeq 0 \end{aligned}$$

This coincides with the standard formulation of the SoS relaxation, 1.6. We can also rewrite the SoS relaxation more explicitly as a semidefinite program, preceding as in Section 1.3.7:

$$\begin{aligned} &\textit{Dual of } (1.9)\textit{:} \quad \text{Find:} \quad && f_{\text{SoS},d}^* - f_0 = \sup_{X^{(i)} \succcurlyeq 0} \sum_{i=0}^r \operatorname{tr}\left(X^{(i)} C_{N_i,0}^{(i)} \right) && (1.11) \\ &\text{subject to:} && \sum_{i=0}^r \operatorname{tr}\left(X^{(i)} C_{N_i,\alpha}^{(i)} \right) - f_\alpha = 0, \quad 0 \neq |\alpha| \leq 2d \\ &&& X^{(0)} \succcurlyeq 0, X^{(1)} \succcurlyeq 0, \ldots, X^{(r)} \succcurlyeq 0 \end{aligned}$$

One could reduce the semidefinite programs above to the canonical form introducing the block diagonal matrices $\operatorname{diag}(H_\Lambda^d, H_{g_1 \star \Lambda}^{N_1}, \ldots, H_{g_r \star \Lambda}^{N_r})$, $\operatorname{diag}(X^{(0)}, X^{(1)}, \ldots, X^{(r)})$ and finally $\operatorname{diag}(C_{d,\alpha}^{(0)}, C_{N_1,\alpha}^{(1)}, \ldots, C_{N_r,\alpha}^{(r)})$.

Remark. The difference between $\overline{\mathcal{Q}_{2d}(\mathbf{g})}$ (obtained in 1.8) and $\mathcal{Q}_{2d}(\mathbf{g})$ (obtained in 1.10 or 1.11) could be explained as follows. While in 1.8 we dualize directly $\mathcal{L}_{2d}(\mathbf{g})$, in 1.10 we dualize the positive semidefinite constraints $H_{g_1 \star \Lambda}^{N_i} \succcurlyeq 0$, we obtain $r+1$ positive semidefinite matrices $X^{(0)}, \ldots X^{(r)}$, and then we consider $\mathcal{Q}_{2d}(\mathbf{g})$ as the image of the map $\phi \colon (X^{(0)}, X^{(1)}, \ldots, X^{(r)}) \mapsto \sum_{i=0}^r g_i \mathbf{b}_{N_i}^t X^{(i)} \mathbf{b}_{N_i}$. The Minkowski sum of closed convex cones may be not closed, and then the image of ϕ may be not closed as well: this is the reason we have in one case $\overline{\mathcal{Q}_{2d}(\mathbf{g})}$ and in the other $\mathcal{Q}_{2d}(\mathbf{g})$.

Chapter 2

The Effective Putinar's Positivstellensatz

This chapter is based on [BM22b] and [BMP22].

2.1 Context and results

A fundamental question in Real Algebraic Geometry is how to represent effectively the set of polynomials which are positive [1] on a given domain. In this chapter, we are going to investigate *exact* effective representations of strictly positive polynomials on *basic closed semialgebraic sets* $S = \mathcal{S}(\mathbf{g}) = \mathcal{S}(g_1,\dots,g_r) = \{x \in \mathbb{R}^n \mid g_i(x) \geq 0 \text{ for } i = 1,\dots,r\}$, defined by inequalities $g_i \geq 0$ with $g_i \in \mathbb{R}[\mathbf{x}]$. These exact representations give immediately an *approximate* representation of nonnegative polynomials on S.

The set $\mathrm{Pos}(S)$ of positive polynomials on S contains the *quadratic module* $Q = \mathcal{Q}(\mathbf{g})$ generated by the tuple of polynomials $\mathbf{g} = (g_1,\dots,g_r)$, and also the *preordering* $O = \mathcal{O}(g_1,\dots,g_r)$ (see Section 1.1.6). Since for a general tuple $\mathbf{g}$ and $n > 1$, positive polynomials on $S(\mathbf{g})$ do not all belong to the quadratic module $Q(\mathbf{g})$ or even to the preordering $\mathcal{O}(\mathbf{g})$, it is natural to ask whether the convex cone $Q(\mathbf{g})$ (resp. $\mathcal{O}(\mathbf{g})$) is a good inner approximation of $\mathrm{Pos}(S)$. A partial answer is given by two important results due to Schmüdgen and Putinar, see Section 1.1.6. These results show the existence of denominator free representations of strictly positive polynomials as elements of the preordering (resp. the quadratic module), in the compact (resp. Archimedean) case.

Since a positive polynomial $f \in \mathrm{Pos}(S)$ can be approximated by $f + \varepsilon$, which is strictly positive on S for $\varepsilon > 0$, these results show that any $f = \lim_{\varepsilon\to 0} f + \varepsilon$ positive on S is the limit of polynomials in $Q(\mathbf{g})$ (resp. $\mathcal{O}(\mathbf{g})$). Unfortunately, the degree of the representation of $f + \varepsilon$ in $\mathcal{Q}(\mathbf{g})$ goes to infinity as $\varepsilon \to 0$, see for instance [Ste96].

In this chapter, we provide quantitative versions of Putinar's Positivstellensatz (Theorem 1.1.22). We give new bounds on the degree of the SoS representation in $Q(\mathbf{g})$, which control the quality of approximation of positive polynomials by polynomials in $Q(\mathbf{g})$. In other words, we consider the *truncated quadratic module* in degree ℓ, denoted $\mathcal{Q}_\ell(\mathbf{g})$ (that is, the convex cone whose elements are polynomials in $\mathcal{Q}(\mathbf{g})$ generated in degree $\leq \ell$, see Section 1.3.7), and given $f > 0$ on $\mathcal{S}(\mathbf{g})$, we estimate $\ell \in \mathbb{N}$ such that $f \in \mathcal{Q}_\ell(\mathbf{g})$. For this problem,

[1] We follow the French tradition, and call a function f *positive* or *nonnegative* on a domain D if $f \geq 0$ on D and *strictly positive* on D if $f > 0$ on D.

also known as *Effective Putinar's Positivstellensatz*, our main result is Theorem 2.2.14, which provides the first polynomial bounds in the intrinsic parameters associated to $\mathbf{g}$ and f. The most important one is $\varepsilon(f) := f^*/\|f\|$, where f^* is the minimum of f on S and $\|f\|$ denote the max norm of f on $[-1,1]^n$: $\varepsilon(f)$ measures how close is f to have a zero on S.

Theorem 2.2.14. *Assume $n \geq 2$ and let $g_1,\dots,g_r \in \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1,\dots,x_n]$ satisfying the normalization assumption* (2.1)*. Let $f \in \mathbb{R}[\mathbf{x}]$ such that $f^* = \min_{x\in S} f(x) > 0$. Let $\mathfrak{c}$, Ł be the Łojasiewicz coefficient and exponent given by Definition 2.2.8. Then $f \in \mathcal{Q}_\ell(\mathbf{g})$ if*

$$\begin{aligned}\ell &\geq O(n^3 2^{5nŁ} r^n \mathfrak{c}^{2n} d(\mathbf{g})^n d(f)^{3.5nŁ} \varepsilon(f)^{-2.5nŁ}) \\ &= \gamma(n,\mathbf{g})\, d(f)^{3.5nŁ} \varepsilon(f)^{-2.5nŁ},\end{aligned}$$

where $\gamma(n,\mathbf{g}) \geq 1$ depends only on n and $\mathbf{g}$.

All the parameters of the bound are explicit, introduced precisely in Section 2.1.3 and Definition 2.2.8. This result is the first general effective Putinar's Positivstellensatz with a polynomial dependence on $\varepsilon(f)$: see Section 2.1.1 for a comparison with the previous results.

We also remark that exponents in Theorem 2.2.14 have been simplified for the sake of readability and are not optimal: see Equation (2.26) for sharper bounds, especially for the case $n \gg 0$. Moreover, that the assumption $n \geq 2$, only used to do this simplification, is not a serious limitation since the univariate case is already well studied, see for instance [PR00].

In particular, the exponent of $\varepsilon(f)^{-1}$ in our result depends on the *Łojasiewicz* exponent Ł between the semialgebraic and algebraic distance from S, see Definition 2.2.8. The study of Łojasiewicz inequalities, already present in [NS07], is a key ingredient for the proof of Theorem 2.2.14. We are able to prove that the Łojasiewicz exponent is equal to one under regularity condition, usually assumed in polynomial optimization problems, improving thus Theorem 2.2.14. This result is presented in Theorem 2.3.9 and Theorem 2.3.13. In the first theorem, we also estimate the Łojasiewicz constant using two parameters: the smallest singular value of the Jacobian of the active constraints at every point of the boundary of S, and a measure of how close is $\mathcal{S}(\mathbf{g})$ to have an extra connected component. In the second theorem, the previous estimate for the Łojasiewicz constant is interpreted as a distance of $\mathbf{g}$ from singular systems of inequalities. To the best of our knowledge, these are the first analysis in the literature of the Łojasiewicz exponent in regular cases and the first estimates for the Łojasiewicz constant.

The main motivation to study the effective Putinar's Positivstellensatz are convergence rates of Lasserre's hierarchies in polynomial optimization (see Section 1.6). Indeed, it is well known that one can determine convergence rates of Lasserre's sum of squares and moment hierarchies using this result. While an exponential dependence on $\varepsilon(f)^{-1}$ in the effective Putinar's Positivstellensatz leads to a *logarithmic* convergence of the lower bound to the minimum, the polynomial dependency on $\varepsilon(f)^{-1}$ in Theorem 2.2.14 gives a *polynomial* convergence of the lower bounds. This general result was not yet proven in the literature, and it is presented in Theorem 2.4.3. For the proof of this result, we observe also that Theorem 2.2.14 gives a quantitative inner approximation of positive polynomials using polynomials in the quadratic module, with degree bounds. This result is described in Theorem 2.4.1.

On the dual side, the dual cones of truncated quadratic modules are outer approximations of the cones of measures, and a section of this dual cone defines the feasible positive linear functionals of moment relaxations. We use Theorem 2.2.14, quantifying how good is the inner approximation of positive polynomials by truncated quadratic modules, to answer the dual question we are interested in: how good is the outer approximation of (probability) measures by truncated positive linear functionals (of total mass one)? We investigate this dual problem, and in particular in Theorem 2.5.9 we bound the Hausdorff distance $\mathrm{d_H}(\cdot,\cdot)$ between the outer approximation and the measures supported on S, providing the first general convergence rate for these outer approximations. As an intermediate step, we also bound the convergence rate to measures not necessarily of mass one (Theorem 2.5.7).

2.1.1 Related works

Complexity analysis in Real Algebraic Geometry is an active area of research, where obtaining good upper bounds is challenging. See for instance [LPR20] for elementary recursive degree bounds in Kivrine-Stengle Positivstellensatz, and [SEDYZ18] for computation complexity of real radicals. Among all the Stellensätzen, we consider Putinar's Positivstellensatz, which allows a denominator free representation of strictly positive polynomials and has well-know applications in Polynomial Optimization.

The representation of strictly positive polynomials has a long history. For instance Pölya [Pó28] gave a representation of homogeneous polynomials f strictly positive on the simplex Δ as ratio of a polynomial with positive coefficients and $\|\mathbf{x}\|_2^k$, for some k. It is interesting to notice that, although no explicit degree bounds were presented, the degree of the representation depends on the sup norm of f on Δ and on its minimum $f^* > 0$, i.e. on $\varepsilon(f)$ (in analogy with our main result, Theorem 2.2.14). This dependence on $\varepsilon(f)$ was made explicit in [PR01]. Another representation for homogeneous polynomials has been proved by Reznick [Rez95], where it is shown that an homogeneous polynomial f strictly positive on $\mathbb{R}^n \setminus \{0\}$ (a *positive definite form*) can be written as ratio of even powers of linear forms and $\|\mathbf{x}\|_2^k$, for some k. Degree bounds for the representation are provided, and again there is a dependence on $\varepsilon(f)$ (defined restricting f to the $n-1$ hypersphere) with exponent equal to -1.

Effective, general versions of Schmüdgen and Putinar's Positivstellensatz, that give degree bounds for the representation in truncaded preorderings and quadratic modules, have been proven by Schweighofer and Nie. In the following, we denote $\|f\|_{\mathbf{x}}$ the max norm of the coefficients of the polynomial f w.r.t. the weighted monomial basis $\{\frac{|\alpha|!}{\alpha_1!\ldots\alpha_n!}\mathbf{x}^\alpha : |\alpha| \le d\}$.

Theorem 2.1.1 ([Sch04]). *For all $\mathbf{g} = g_1,\ldots g_r \subset \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1,\ldots,x_n]$ defining $\emptyset \neq \mathcal{S}(\mathbf{g}) = S \subset (-1,1)^n$ there exists $0 < c \in \mathbb{R}$ (depending on $\mathbf{g}$ and n) such that, if $f \in \mathbb{R}[\mathbf{x}]_d$ is strictly positive on S with minimum $f^* = \min_{x\in S} f(x) > 0$, we have $f \in \mathcal{O}_\ell(\mathbf{g})$ if*

$$\ell \ge cd^2\Big(1 + \Big(d^2 n^d \frac{\|f\|_{\mathbf{x}}}{f^*}\Big)^c\Big).$$

Theorem 2.1.2 ([NS07]). *For all $\mathbf{g} = g_1,\ldots g_r \subset \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1,\ldots,x_n]$ defining an Archimedean quadratic module $Q = \mathcal{Q}(\mathbf{g})$ and $\emptyset \neq \mathcal{S}(\mathbf{g}) = S \subset (-1,1)^n$, there exists $0 < c \in \mathbb{R}$ (depending on $\mathbf{g}$ and n) such that, if $f \in \mathbb{R}[\mathbf{x}]_d$ is strictly positive on S with minimum $f^* = \min_{x\in S} f(x) > 0$, we have $f \in \mathcal{Q}_\ell(\mathbf{g})$ if*

$$\ell \ge c\exp\bigg(\Big(d^2 n^d \frac{\|f\|_{\mathbf{x}}}{f^*}\Big)^c\bigg).$$

In Theorem 2.2.14 we improve Theorem 2.1.2, showing a polynomial (and not exponential) dependence of the degree of the representation on $\varepsilon(f)^{-1} = \|f\|/f^*$. The norm $\|\cdot\|_{\mathbf{x}}$ used in [Sch04] and [NS07] is the max norm of the coefficients of the polynomial f w.r.t. the weighted monomial basis $\{\frac{|\alpha|!}{\alpha_1!\ldots\alpha_n!}\mathbf{x}^\alpha : |\alpha| \leq d\}$, while the one we will use is the max norm on $[-1,1]^n$, see Section 2.1.3. In the definition of $\varepsilon(f)$ we use the max norm $\|\cdot\|$ on $[-1,1]^n$ instead of $\|\cdot\|_{\mathbf{X}}$ used in [NS07], because it does not depend on the choice of a basis and on the representation of the polynomials. However, for polynomials of bounded degree, the two norms are equivalent. Using [NS07, lem. 7] to express our bound with $\|\cdot\|_{\mathbf{x}}$ would result in an extra factor $2^{2.5nŁ}n^{2.5d(f)nŁ}d(f)^{2.5nŁ}$, while keeping the exponent of $\varepsilon(f)$.

Apart from the already mentioned corollary on the convergence rates for Lasserre's hierarchies, this result is also used in [MSED21] to give bounds on the degree of rational SoS positivity certificates, which are exponential in the bit-size of the input polynomials $f, \mathbf{g}$.

As already mentioned, compared to [NS07], Theorem 2.2.14 gives degree bounds, which are polynomial and not exponential in $\varepsilon(f)$. This implies a polynomial rate convergence of Lasserre hierarchies, see Theorem 2.4.2, and not logarithmic as in [NS07]. For special semialgebraic sets, the bounds on the convergence rate can be improved: see for instance [FF20] for convergence in the unit sphere, [LS21] for the unit box, and [Slo21] for convergence on the unit ball. The convergence rate of the upper bounds of Lasserre SoS density hierarchy over the sphere has been studied in [DKL19].

The proof of Theorem 2.2.14 is based on the construction of a perturbation polynomial $q \in \mathcal{Q}(\mathbf{g})$ and the reduction to a simpler semialgebraic set. This construction of the perturbation polynomial q using univariate SoS, has already been used in [Sch04], [Sch05c], [NS07], [Ave13], [KS15]. In [MM22] Mai and Magron investigate with a similar technique the representation of strictly positive polynomials on arbitrary semialgebraic sets as ratio of polynomials in the quadratic module and $(1 + \|\mathbf{x}\|_2^2)^k$ for some k, giving degree bounds for the representation. These bounds are polynomial on f^* (and thus on $\varepsilon(f)$), but the exponent and the constant are not explicit in the general case. Moreover, they remark that they were not able to derive a polynomial Effective Putinar's Positvstellensatz using their perturbation polynomials, defined recursively.

Our main improvements in the proof are the generalisation from a special univariate SoS or recursively defined perturbation polynomials, to a SoS approximation of a positive plateau function, see Section 2.2.4, and the use of an Effective Schmüdgen's Positivstellensatz on the unit box from [LS21]. Moreover, in Section 2.3 we analyze regular cases that result in very simple exponents, see Corollary 2.3.14. Corollary 2.3.14 can be applied in particular in the case of a single ball constraint, that was analyzed in [MM22] for the Putinar-Vasilescu's Positivstellensatz, that introduces a denominator: the exponent in this case is equal to -65, while Corollary 2.3.14 gives $-2.5n$. We conjecture that it is also possible to remove the dependence on n in the exponent of the effective Putinar's Positivstellensatz.

This approach with a perturbation polynomial q has also been used in [KS15] to prove a Weierstrass Approximation theorem on compact sets for positive polynomials, where the approximation is done with polynomials in the quadratic module $\mathcal{Q}(\mathbf{g})$. We obtain an equivalent result with our polynomial echelon functions in Theorem 2.4.1 with bounds on the degree of q.

Convergence of pseudo-moments sequences to measures in Lasserre's hierarchies has been studied in [Sch05c] for Polynomial Optimization Problems and more generally in [Tac21] for Generalized Moment Problems (GMP). The convergence rates of moment hierarchies in GMP over the simplex and the sphere have been studied in [KK21]. To our best knowledge there is no analysis of the convergence rate for general compact basic semialgebraic sets in the literature. In Theorem 2.5.9 we prove such a rate of convergence for the pseudo-moment sequences used in polynomial optimization, deducing this speed from Theorem 2.2.14.

2.1.2 Structure of the chapter

The chapter is organized as follows.

- Section 2.1 describes the contributions and provide context for the chapter. In Section 2.1.1 the relevant literature is presented, and our results are compared to already existing contributions. Section 2.1.2 describes the organization of the chapter, and finally Section 2.1.3 presents the most important notations of the chapter.

- Section 2.2 is devoted to the proof of Theorem 2.2.14. We explain the general idea of the proof (Section 2.2.1) and show how to reduce the proof from general semialgebraic sets to simpler domains. The general principles for this reduction are clarified in Section 2.2.2, while in Section 2.2.3 the Łojasiewicz inequalities needed for this quantitative investigation are introduced. The reduction to simpler domains is completed in Section 2.2.4, where we construct a SoS polynomial using polynomial Urysohn functions. We conclude the proof of Theorem 2.2.14 in Section 2.2.5.

- Section 2.3 investigate a Łojasiewicz inequality under a regularity assumption, namely constraint qualification condition. In Section 2.3.1 we study local properties of the distance and semialgebraic distance functions from S, using first order Taylor expansions. In Section 2.3.2 show that the Łojasiewicz exponent is equal to one under regularity conditions, and give different estimates for the Łojasiewicz constant.

- Section 2.4 analyzes the convergence for the optima of the Lasserre's hierarchies. We prove the first general polynomial convergence of these optima, as a corollary of the effective Putinar's Positivstellensatz, interpreting this result as a quantitave version of Weierstrass approximation theorem.

- Section 2.5 describes the dual problem of approximation of positive polynomials, namely the approximation of measures using positive linear functionals. We first bound the Hausdorff distance of normalized positive linear functionals to measures, and then bound the distance to probability measures in Section 2.5.1.

- Section 2.6 concludes the chapter, describing possible directions of research: the improvement of the bound that we obtained (Section 2.6.1), a generalization of Putinar's Positivstellensatz and Lasserre's hierarchies (Section 2.6.2), applications to the generalized moment problem (Section 2.6.3) and finally the deduction of certificates of emptiness for basic closed semialgebraic sets (Section 2.6.4).

2.1.3 Notations and assumptions

We denote $\|\mathbf{x}\|_2^2 = x_1^2 + \cdots + x_n^2 \in \mathbb{R}[x_1, \ldots, x_n] = \mathbb{R}[\mathbf{x}]$. We recall that a quadratic module $\mathcal{Q}(\mathbf{g})$ is called *Archimedean* if $r^2 - \|\mathbf{x}\|_2^2 \in \mathcal{Q}(\mathbf{g})$ for some $r \in \mathbb{R}_{>0}$, see Definition 1.1.20. However, to simplify the proofs we assume that $r = 1$:

$$\textbf{Normalization assumption}: 1 - \|\mathbf{x}\|_2^2 \in \mathcal{Q}(\mathbf{g}). \tag{2.1}$$

We can always be in this setting by a change of variables if we start with an Archimedean quadratic module: if $r^2 - \|\mathbf{x}\|_2^2 \in \mathcal{Q}(\mathbf{g})$ then $1 - \|\mathbf{x}\|_2^2 \in \mathcal{Q}(\mathbf{g}(r\mathbf{x}))$ (i.e. the quadratic module generated by $g_i(rx_1, \ldots, rx_n)$). Notice also that the normalization assumption implies that $S \subset \mathcal{S}(1 - \|\mathbf{x}\|_2^2)$.

We list hereafter the most important notations that we will need through the chapter:

- $f \in \mathbb{R}[\mathbf{x}]$ is a polynomial in n variables of degree $d = d(f) = \deg f$;
- $S = \mathcal{S}(\mathbf{g}) = \mathcal{S}(g_1, \ldots, g_r)$ is the basic closed semialgebraic set defined by $\mathbf{g} = g_1, \ldots, g_r$;
- $D \supset S$ a simple compact domain containing the unit ball (and thus S, from (2.1));
- $d(\mathbf{g}) = \max_i \deg g_i$ is the maximum degree of the inequalities defining S;
- $f^* = \min_{x \in S} f(x)$ is the minimum of f on S, and unless otherwise stated $f^* > 0$;
- $\|\cdot\|$ denotes the max norm of a polynomial on D, i.e. $\|h\| = \max_{x \in D} |h(x)|$;
- $\varepsilon(f) = \frac{f^*}{\|f\|}$ is a measure of how close is f to have a zero on S.

2.2 Proof of the effective Putinar's Positivstellensatz

In this section, we develop the proof of Theorem 2.2.14.

2.2.1 Idea of the proof

Putinar's Positivstellensatz gives a representation of a *strictly* positive polynomial on $\mathcal{S}(\mathbf{g})$ as an element of the Archimedean quadratic module $\mathcal{Q}(\mathbf{g})$, and this shows that $\mathcal{Q}(\mathbf{g})$ is a good inner approximation of $\mathrm{Pos}(\mathcal{S}(\mathbf{g}))$, the convex cone of the nonnegative (or positive) polynomials on $\mathcal{S}(\mathbf{g})$: $\overline{\mathcal{Q}(\mathbf{g})} = \mathrm{Pos}(\mathcal{S}(\mathbf{g}))$. But why this is the case? We give an answer based on a (multivariate) approximation theory, that will be the idea behind the proof of the Effective Putinar's Positivstellensatz.

Working with arbitrary semialgebraic sets $\mathcal{S}(\mathbf{g})$ is difficult: therefore, we try to reduce the Proof of Putinar's Positivstellensatz to simpler cases, where we have good properties of approximation of positive functions. The best we can hope are (convex) compact domains defined by affine inequalities, where we have good approximation properties of positive polynomials. In particular, there are good approximation properties:

- for hypercubes, where we can use properties of the Bernstein basis [KL10] or polynomial kernels [LS21] to deduce effective version of Schmüdgen's Positivstellensatz;
- for simplexes, where we have effective versions of Polya's theorem [PR01].

We show in the following that we can reduce to these cases, for Archimedean quadratic modules.

Lemma 2.2.1. *Let $L(\mathbf{a}) = a_0 + a_1 x_1 + \cdots + a_n x_n \in \mathbb{R}[\mathbf{x}]$ be an affine polynomial such that $L(\mathbf{a}) \geq 0$ on $\mathcal{S}(1 - \|\mathbf{x}\|_2^2)$. Then $L(\mathbf{a}) \in \mathcal{Q}(1 - \|\mathbf{x}\|_2^2)$.*

Moreover, if $L(\mathbf{a}_1), \ldots, L(\mathbf{a}_r)$ are affine polynomials such that $\mathcal{S}(L(\mathbf{a}_1), \ldots, L(\mathbf{a}_r)) \supset \mathcal{S}(1 - \|\mathbf{x}\|_2^2)$, then the preordering generated by $L(\mathbf{a}_1), \ldots, L(\mathbf{a}_r)$ is contained in $\mathcal{Q}(1 - \|\mathbf{x}\|_2^2)$, and precisely we have:

$$\mathcal{O}_d(L(\mathbf{a}_1), \ldots, L(\mathbf{a}_r)) \subset \mathcal{Q}_{d+r}(1 - \|\mathbf{x}\|_2^2)$$

Proof. Let $L(\mathbf{a})$ be as in the hypothesis. Since $1 - \|\mathbf{x}\|_2^2$ is invariant under the action of the orthogonal group, we can assume that $L(\mathbf{a}) = r + x_i$, $r \geq 1$. Then notice that:

$$\begin{aligned} L(\mathbf{a}) &= (r-1) + (1+x_i) = (r-1) + \frac{1}{2}((1 - x_i^2 + (1+x_i)^2)) \\ &= (r-1) + \frac{1}{2}((1 - \|\mathbf{x}\|_2^2 + \sum_{j \neq i} x_j^2 + (1+x_i)^2)) \in \mathcal{Q}_2(1 - \|\mathbf{x}\|_2^2). \end{aligned} \tag{2.2}$$

Assume now $\mathcal{S}(L(\mathbf{a}_1), \ldots, L(\mathbf{a}_r)) \supset \mathcal{S}(1 - \|\mathbf{x}\|_2^2)$. The previous point implies that

$$\mathcal{Q}_d(L(\mathbf{a}_1), \ldots, L(\mathbf{a}_r)) \subset \mathcal{Q}_{d+1}(1 - \|\mathbf{x}\|^2).$$

We need to show that products of the $L(\mathbf{a}_i)$ are in $\mathcal{Q}_{d+1}(1 - \|\mathbf{x}\|^2)$ as well. But notice that $\mathcal{Q}(1 - \|\mathbf{x}\|^2)$ is a preordering (i.e. it is closed under multiplication) since it is generated by a single polynomial: therefore $\mathcal{O}(L(\mathbf{a}_1), \ldots, L(\mathbf{a}_r)) \subset \mathcal{Q}(1 - \|\mathbf{x}\|_2^2)$ follows. More precisely, taking products of equations of the form (2.2) we have:

$$\mathcal{O}_d(L(\mathbf{a}_1), \ldots, L(\mathbf{a}_r)) \subset \mathcal{Q}_{d+r}(1 - \|\mathbf{x}\|_2^2)$$

concluding the proof of the lemma. □

Lemma 2.2.1 shows that, if we have a polyhedron defined by affine inequalities and an approximation property of positive polynomials for the preordering defined by these affine inequalities, then the same approximation property holds for the quadratic module generated by any ball contained in the polyhedron. In particular this is true in the case:

- of the unit box $[-1,1]^n$, where we have: $\mathcal{O}_d(1 \pm x_i : i \in \{1, \ldots, n\}) \subset \mathcal{Q}_{d+n}(1 - \|\mathbf{x}\|_2^2)$;
- for the (scaled, translated) simplex

  $$\Delta_n = \{x \in \mathbb{R}^n \mid 1 + x_1 \geq 0, \ldots, 1 + x_n \geq 0, n\sqrt{n} - x_1 - \cdots - x_n \geq 0\},$$

  where we have $\mathcal{O}_d(1 + x_1, \ldots 1 + x_n, n\sqrt{n} - x_1 - \cdots - x_n) \subset \mathcal{Q}_{d+n+1}(1 - \|\mathbf{x}\|_2^2)$

It is also important to remark that we are not using Schmüdgen's or Putinar's Positivstellensatz for $\mathcal{Q}(1 - \|\mathbf{x}\|_2^2)$ or $\mathcal{Q}(\mathbf{g})$ to prove Lemma 2.2.1.

The discussion above shows that, if we can reduce the problem to one of these simpler domains D, Putinar's Positivstellensatz follows from the approximation theorem for positive polynomials on D. Concretely, to reduce to this simpler domain we proceed as follows, refining the approach in [Sch05c; NS07; Ave13]:

(i) we perturb f into a polynomial $p = f - q$ such that p is strictly positive on D and $q = f - p$ is in the quadratic module $\mathcal{Q}(\mathbf{g})$;

(ii) we have a representation of p in $\mathcal{Q}(1 - \|\mathbf{x}\|_2^2) \subset \mathcal{Q}(\mathbf{g})$, using the approximation theorem on D and Lemma 2.2.1;

(iii) finally we deduce the representation $f = p + q \in \mathcal{Q}(\mathbf{g})$.

While the third point is trivial and the second point can be deduced from existing results and Lemma 2.2.1, the first point is the key one: the next sections will be devoted to the construction of p with degree bounds for its representation as an element of $\mathcal{Q}(\mathbf{g})$. It will be clear that also the construction of q depends on approximation properties of positive polynomials.

Notice that if $f > 0$ on D, then we can skip the first point and conclude immediately the proof. Therefore, in the following we always assume that there exists $x \in D \setminus S$ such that $f(x) \leq 0$.

2.2.2 Reduction to simpler domains: general principles

We start describing general principles to construct the polynomial q, that is used to reduce the proof from $\mathcal{S}(\mathbf{g})$ to D, as mentioned in the previous section.

Let $f > 0$ on $S = \mathcal{S}(\mathbf{g})$, S compact, and denote $f^* > 0$ the minimum of f on S. Let $D \supset S$ be another compact domain that we will specify in Section 2.2.5. Our goal is to find $q \in \mathcal{Q}(\mathbf{g})$ such that $p = f - q > 0$ on D. More precisely, we need the following bounds:

B.1 the minimum of p on D has to be of the same order of f^*, say $p = f - q \geq \frac{f^*}{2}$ on D;

B.2 bounds on the degree m such that $q \in \mathcal{Q}_m(\mathbf{g})$ (or, equivalently, on $\deg p$);

B.3 bounds on $\|p\|$.

Furthermore, notice that we have $q \geq 0$ on S since $q \in \mathcal{Q}(\mathbf{g})$.

We abstract temporarily the setting: we ignore degree and norm bounds, and we remove the restriction to polynomials as well: we look for a continuous function u on D such that $f - u \geq \frac{f^*}{2}$ on D and $u \geq 0$ on S. To find such a u, it is natural to partition D using the sublevel sets of f, for instance considering:

$$A := \{x \in D \mid f(x) \leq \frac{3f^*}{4}\}.$$

Recall that we are assuming that there exists $x \in D \setminus S$ such that $f(x) \leq 0$, so that $A \neq \emptyset$. Furthermore, notice that $A \cap S = \emptyset$ since $f \geq f^*$ on S and $f \leq \frac{3f^*}{4}$ on A. Then we observe the following:

- On S, u has to be small, say $0 \leq u(x) \leq \frac{f^*}{2}$ for all $x \in S$, so that $f - u \geq \frac{f^*}{2}$ on S.
- On A, u has to be negative, say $u(x) \leq -\|f\| - \frac{f^*}{2}$, so that $f - u \geq \frac{f^*}{2}$ on A.

The existence of such a continuous function u is guaranteed from Urysohn's Lemma.

Theorem 2.2.2 (Urysohn's Lemma, [Mun00, th. 33.1]). *Let X be a normal topological space (i.e. a topological space where any two disjoint closed sets can be separated by two disjoint open sets). Let C_1, C_2 be disjoint closet sets and $[a,b] \subset \mathbb{R}$. Then there exists a continuous function $u\colon X \to [a,b]$ such that $u = a$ on C_1 and $u = b$ on C_2.*

Therefore, from Theorem 2.2.2 there exists an Urysohn function $u\colon D \to [-\|f\| - \frac{f^*}{2}, \frac{f^*}{2}]$ such that $u(x) = \frac{f^*}{2}$ for all $x \in S$ and $u(x) = -\|f\| - \frac{f^*}{2}$ for all $x \in A$. Moreover, it is clear that

$f - u \geq \frac{f^*}{2}$ on D and $u \geq 0$ on S, as we wanted. Since we are in a metric space, a possible explicit expression for u is:

$$u(x) = -\big(\|f\| + f^*\big)\frac{\mathrm{d}(x,S)}{\mathrm{d}(x,S)+\mathrm{d}(x,A)} + \frac{f^*}{2}$$

where $\mathrm{d}(x,Y)$ denotes the Euclidean distance of x from Y. See also Section 2.2.3 for more on the Euclidean distance function.

To solve our initial problem, we want to replace the continuous function u with a polynomial $q \in \mathcal{Q}(\mathbf{g})$ that shares (approximately) the same properties of u, i.e. we want to find a *polynomial approximation* $q \in \mathcal{Q}(\mathbf{g})$ of u. We refer then to q as a *polynomial Urysohn function*.

Write $q = s\sum_i s_i \frac{g_i}{\|g_i\|} \in \mathcal{Q}(\mathbf{g})$ for some $s \in \mathbb{R}_{>0}$ and $s_i \in \Sigma^2$ (the normalization from g_i to $\frac{g_i}{\|g_i\|}$ and the parameter $s \in \mathbb{R}_{>0}$ will be convenient later). As in the case of u, we want the polynomial Urysohn function q to be sufficiently negative on A, and positive with the same order of f^* on S. Furthermore, recall that $A \cap S = \emptyset$, and thus for all $x \in A$ there exists $i \in \{1,\dots r\}$ such that $g_i(x) < 0$. Since A is compact, there exists $\delta \in \mathbb{R}_{>0}$ such that:

$$\min\{g_1(x),\dots g_r(x)\} \leq -\delta \text{ for all } x \in A. \tag{2.3}$$

The parameter δ depends on A and $\mathbf{g}$, i.e. on how f and $\mathbf{g}$ vary. It will be the main parameter for the construction of the SoS coefficients s_i of q, and thus it will be the key ingredient to obtain the bounds **B.1**, **B.2** and **B.3**. Indeed:

(i) if $\frac{g_i(x)}{\|g_i\|} \leq -\delta$ implies that $s_i(x)$ is approximately one, then $s\sum_i s_i(x)\frac{g_i(x)}{\|g_i\|}$ will be negative enough for $x \in A$, when $s \in \mathbb{R}_{>0}$ is big enough;

(ii) if $\frac{g_i(x)}{\|g_i\|} \geq 0$ implies that $s_i(x)$ is small enough, then $s\sum_i s_i(x)\frac{g_i(x)}{\|g_i\|}$ will be small enough for $x \in S$.

In Section 2.2.3, we analyze and bound the parameter δ in terms of f and $\mathbf{g}$ using appropriate Łojasiewicz inequalities. In Section 2.2.4 we effectively construct the polynomial Urysohn function $q \in \mathcal{Q}(\mathbf{g})$ following the ideas above, bounding its norm and the degree of its representation as an element of the quadratic module $\mathcal{Q}(\mathbf{g})$. Finally, in Section 2.2.5 we conclude the proof of the Effective Putinar's Positivstellensatz.

2.2.3 Reduction to simpler domains: Łojasiewicz inequalities

We have seen in Section 2.2.2 that, for the effective construction of the polynomial Urysohn function q, we need to compare on D the behavior of the function f with the behavior of the functions $g_1,\dots,g_r$. In particular, we will relate δ (introduced in (2.3)) with $\varepsilon(f) = \frac{f^*}{\|f\|}$ in Proposition 2.2.9. To do that, we will need to consider some *continuous semialgebraic functions*.

Definition 2.2.3. We say that $H\colon \mathbb{R}^n \to \mathbb{R}$ is a *continuous semialgebraic function* if it is continuous with respect to the Euclidean topology and its graph is a semialgebraic set in $\mathbb{R}^{n+1}$.

We introduce the two continuous semialgebraic functions that will be central to compare f and $\mathbf{g}$ on D. For $x \in D$, let

$$F(x) := -\min\left(\frac{f(x)-f^*}{\|f\|}, 0\right) \tag{2.4}$$

$$G(x) := -\min\left(\frac{g_1(x)}{\|g_1\|}, \dots, \frac{g_s(x)}{\|g_s\|}, 0\right). \tag{2.5}$$

The function G can be described as a *semialgebraic distance to S*, since $x \in S$ if and only if $G(x) = 0$. Using the language of error bounds in optimization, the function G can be also considered as a *residual function*, see [Pan97]. Moreover, if δ is as in (2.3):

$$\forall x \in A,\ G(x) \geq \delta \tag{2.6}$$

and δ can be described precisely as a lower bound for the minimum of G on A.

To show how δ depends on $\varepsilon(f)$, we use Łojasiewicz inequalities, introduced by Łojasiewicz in [Łoj59], following and expanding the approach in [NS07, lem. 13].

Theorem 2.2.4 ([Łoj59], [BCR98, cor. 2.6.7]). *Let D be a closed and bounded semialgebraic set and let F, G be two continuous semialgebraic functions from D to $\mathbb{R}$ such that $F^{-1}(0) \supset G^{-1}(0)$. Then there exists $c, Ł \in \mathbb{R}_{>0}$ such that $\forall x \in D$:*

$$|F(x)|^{Ł} \leq c|G(x)|.$$

We use now Theorem 2.2.4 to compare F and the semialgebraic distance G (notice that it is possible to apply the theorem in this case, since $F^{-1}(0) = \{x \in D \mid f(x) \geq f^*\} \supset G^{-1}(0) = S$).

Definition 2.2.5. We denote $\mathfrak{c}_1, Ł_1$ the smallest constant and exponent for Łojasiewicz inequalities (Theorem 2.2.4) between the functions F (2.4) and G (2.5): for all $x \in D$,

$$F(x)^{Ł_1} \leq \mathfrak{c}_1\, G(x). \tag{2.7}$$

These constant and exponent are well-defined by Theorem 2.2.4, since the functions F and G are continuous semialgebraic and $S = G^{-1}(0) \subset F^{-1}(0)$.

To analyze these exponent and constant, we first relate F to the *Euclidean distance* function

$$E\colon D \ni x \mapsto E(x) = \mathrm{d}(x, S).$$

This is a continuous semialgebraic function, see for instance [BCR98, prop. 2.2.8]. Recall also that the Łojasiewicz Inequality for the distance function to the zero set of a polynomial or a real analytic function is the original one, introduced in the polynomial case by Hörmander [Hör58] and in the analytic case by Łojasiewicz [Łoj59].

In order to relate F and E, we need a Markov inequality.

Theorem 2.2.6 ([KR99, th. 3]). *Let $p \in \mathbb{R}[\mathbf{x}]_d$ be a polynomial of degree $\leq d$. Then:*

$$\big\| \|\nabla p\|_2 \big\| = \max_{x \in D} \|\nabla p(x)\|_2 \leq \frac{2d(2d-1)}{w(D)} \|p\|$$

where $w(D)$, the width of D, is the minimal distance between two distinct parallel touching hyperplanes.

Proposition 2.2.7. *Let F and G be as above. Then, if the normalization assumption* (2.1) *is satisfied, we have the following Łojasiewicz inequality between F and D:* $\forall x \in D$,

$$F(x) \leq \frac{4d^2 - 2d}{w(D)} E(x) \leq 2d^2 E(x) \tag{2.8}$$

where $w(D)$ *is the width of* D *and* $d = \deg(f)$.

Proof. For $y \in D$ and $z \in S$ such that $E(y) = \mathrm{d}(y, S) = \|y - z\|$, we have

$$F(y) = F(y) - F(z) \leq L_F \|y - z\| = L_F E(y) \leq L_f E(y), \tag{2.9}$$

where L_F is the Lipschitz constant of F on D, L_f is the Lipschitz constant of $\frac{f}{\|f\|}$ on D, and the last inequality follows from $L_F \leq L_f$. From the mean value theorem we deduce that for all $x, y \in D$ we have $\left|f(x) - f(y)\right| \leq \left\|\|\nabla f\|_2\right\| \left\|x - y\right\|_2$. Then from the definition of Lipschitz constant and Theorem 2.2.6:

$$L_f \leq \frac{\left\|\|\nabla f\|_2\right\|}{\|f\|} \leq \frac{2d(2d-1)}{w(D)} \leq 2d^2, \tag{2.10}$$

where the last inequality holds true since $\mathcal{S}(1 - \|\mathbf{x}\|_2^2) \subset D$ implies $w(D) \geq 2$. From (2.9) and (2.10) we finally deduce (2.8). □

To complete the analysis we now need to relate the Euclidean distance E and the semi-algebraic distance G using another Łojasiewicz inequality. This is possible, since $G^{-1}(0) = E^{-1}(0) = S$.

Definition 2.2.8. We denote $\mathfrak{c}, Ł$ the smallest constant and exponent for Łojasiewicz inequalities (Theorem 2.2.4) between the functions E (2.4) and G (2.5): for all $x \in D$,

$$E(x)^{Ł} \leq \mathfrak{c}\, G(x). \tag{2.11}$$

We can now finally describe δ in terms of f and G.

Proposition 2.2.9. *With F and G defined in* (2.4) *and* (2.5), $\mathfrak{c}, Ł$ *defined in Definition 2.2.8 and* $d = \deg(f)$, *we have:*

$$F(x)^{Ł} \leq 2^{Ł} d^{2Ł} \mathfrak{c} G(x)$$

for all $x \in D$. *Moreover, we can choose* $\delta = \frac{1}{\mathfrak{c}}\left(\frac{\varepsilon(f)}{8d^2}\right)^{Ł}$ *in Equation* (2.3) *and Equation* (2.6).

Proof. The inequality $F(x)^{Ł} \leq 2^{Ł} d^{2Ł} \mathfrak{c} G(x)$ follows combining (2.11) and (2.8).

For the second part, let $x \in A$. Then $F(x) = \frac{f^* - f(x)}{\|f\|} \geq \frac{f^*}{4\|f\|}$. Therefore, from the inequality above:

$$\left(\frac{f^*}{4\|f\|}\right)^{Ł} \leq F(x)^{Ł} \leq (2d^2)^{Ł} \mathfrak{c} G(x).$$

This implies that we can choose $\delta = \frac{1}{\mathfrak{c}}\left(\frac{\varepsilon(f)}{8d^2}\right)^{Ł}$ in Equation (2.3) and Equation (2.6). □

Proposition 2.2.9 gives a bound for $\mathfrak{c}_1, Ł_1$ in (2.7) using $\mathfrak{c}, Ł$, d and $\varepsilon(f)$. It would be possible to work directly with $\mathfrak{c}_1, Ł_1$, but we prefer to use $\mathfrak{c}$ and $Ł$, since $\mathfrak{c}$ and $Ł$ are independent of f: indeed, all the dependence on f in the Łojasiewicz inequality in Proposition 2.2.9 is encoded in F and d.

2.2.4 Reduction to simpler domains: polynomial Urysohn functions

In this section, we construct the polynomial Urysohn function $q \in \mathcal{Q}(\mathbf{g})$ used to perturb f to $p = f - q > 0$ on D, so that the bounds **B.1**, **B.2** and **B.3** in Section 2.2.2 are satisfied.

In particular, we describe the SoS coefficients of q using the square of a univariate polynomial Urysohn function $h_{k,m}$. We will call this univariate polynomial Urysohn function $h_{k,m}$ a *plateau polynomial*, to recall its shape (see Figure 2.1) and to distinguish it from the polynomial Urysohn function q. The final expression of q in terms of $h_{k,m}$ will be the following:

$$q(x) := s \sum_{i=1}^{r} h_{k,m}\left(\frac{g_i(x)}{\|g_i\|}\right)^2 \frac{g_i(x)}{\|g_i\|} \tag{2.12}$$

where s, m and k are positive real parameters, to be specified later in Proposition 2.2.11 and Proposition 2.2.12, and $h_{k,m}$ is defined in Proposition 2.2.11. Notice that, since we are taking the square of $h_{k,m}$ and $s \geq 0$, naturally $q \in \mathcal{Q}(\mathbf{g})$. The final estimates that we obtain for $\|q\|$ and $\deg q$ (or, equivalently, for $\|p\|$ and $\deg p$) are presented in Proposition 2.2.13.

Let us now construct the plateau polynomial $h_{k,m}$. This plateau polynomial depends on a parameter $k \in \mathbb{R}_{>0}$ controlling the minimum of the function, and it will be of degree m. Both of these parameters will depend on $\delta \in \mathbb{R}_{>0}$ (defined in (2.6) or (2.3)), and thus in particular on $\varepsilon(f)$ (see Proposition 2.2.9).

The plateau polynomial $h_{k,m}$ will be an approximation of a univariate function $H(t)$, that we define as a smooth Urysohn function for $[-1,-\delta]$ and $[0,1]$ as a piecewise cubic spline:

$$H(t) = \begin{cases} 1 & t \in [-1,-\delta] \\ -\frac{9(k-1)}{2d^3k}t^3 - \frac{27(k-1)}{2d^2k}t^2 - \frac{27(k-1)}{2dk}t - \frac{7k-9}{2k} & t \in [-\delta, -\delta + \frac{\delta}{3}] \\ \frac{9(k-1)}{d^3k}t^3 + \frac{27(k-1)}{2d^2k}t^2 + \frac{9(k-1)}{2dk}t + \frac{k+1}{2k} & t \in [-\delta + \frac{\delta}{3}, -\delta + \frac{2\delta}{3}] \\ -\frac{9(k-1)}{2d^3k}t^3 + \frac{1}{k} & t \in [-\delta + \frac{2\delta}{3}, 0] \\ \frac{1}{k} & t \in [0,1] \end{cases} \tag{2.13}$$

Notice that δ controls the width of the step of $H(t)$, see Figure 2.1. The piecewise polynomial function $H(t)$ is a C^2 cubic spline on $[-1,1]$. Indeed, an explicit computation shows that the functions $H, H^{(1)}, H^{(2)}$ are absolutely continuous, and moreover the piecewise constant function $H^{(3)}$ is of total variation $V = \frac{216(k-1)}{\delta^3 k}$. Finally notice that H is non-increasing on $[-1,1]$.

We are now ready to construct the plateau polynomial $h_{k,m}$, in such a way that $h_{k,m}^2$ is a uniform approximation of H with error $\leq \frac{1}{k}$, and giving estimates for m using k and δ. In particular, in Proposition 2.2.11 we define $h_{k,m}$ as an approximation of H by a polynomial $\in \mathbb{R}[t]$, using Chebyshev approximation (see Figure 2.1):

Theorem 2.2.10 (Chebyshev approximation on $[-1,1]$ [Tre13])**.** *For an integer u, let $h : [-1,1] \to \mathbb{R}$ be a function such that its derivatives through $h^{(u-1)}$ are absolutely continuous on $[-1,1]$ and its u-th derivative $h^{(u)}$ is of bounded variation V. Then its Chebyshev approximation p_m of degree m satisfies:*

$$\|h - p_m\| \leq \frac{4V}{\pi u (m-u)^u}$$

Proposition 2.2.11. *There exists a univariate polynomial $h_{k,m} \in \mathbb{R}[t]$, that we call* plateau polynomial, *such that:*

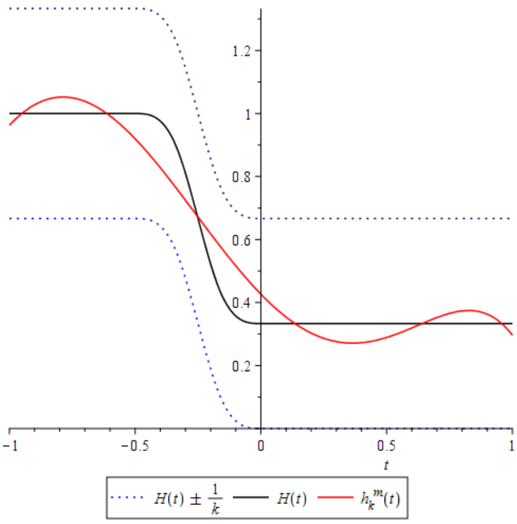

Figure 2.1: Plateau polynomial

- $\deg h_{k,m} = m$ *with* $m = \left\lceil \frac{6}{\delta}\sqrt[3]{\frac{4(k-1)}{\pi}} + 3 \right\rceil$;
- *for* $t \in [-1, -\delta]$ *we have* $1 - \frac{1}{k} \le h_{k,m}(t)^2 \le 1 + \frac{1}{k}$;
- *for* $t \in [0, 1]$ *we have* $h_{k,m}(t)^2 \le \frac{2}{k}$;
- *for* $t \in [-1, 1]$ *we have* $0 \le h_{k,m}(t)^2 \le 1 + \frac{1}{k}$.

Proof. We construct a degree m Chebyshev approximation $h_{k,m} \in \mathbb{R}[t]$ of H such that

$$\left\| H - h_{k,m} \right\| \le \frac{1}{3k} \tag{2.14}$$

As H, $H^{(1)}$ and $H^{(2)}$ are absolutely continuous and $H^{(3)}$ has total variation $V = \frac{216(k-1)}{\delta^3 k}$, by Theorem 2.2.10, it suffices to take m such that $\frac{4V}{3\pi(m-3)^3} \le \frac{1}{3k}$, i.e.

$$m \ge \sqrt[3]{\frac{4Vk}{\pi}} + 3 = \frac{6}{\delta}\sqrt[3]{\frac{4(k-1)}{\pi}} + 3,$$

which proves the first point.

The other points follow from (2.14) and the definition of H in (2.13). For instance, if $t \in [-1, -\delta]$ then $0 \le 1 - \frac{1}{3k} \le h_{k,m}(t) \le 1 + \frac{1}{3k}$ (for the first inequality, k will be selected later to be ≥ 2 later). Therefore:

$$h_{k,m}(t)^2 \ge \left(1 - \frac{1}{3k}\right)^2 = 1 - \frac{2}{3k} + \frac{1}{9k^2} \ge 1 - \frac{2}{3k} \ge 1 - \frac{1}{k}$$

The other points can be proven similarly. □

Proposition 2.2.11 is the first step to get the bounds **B.2** and **B.3** in Section 2.2.2.

Remark. Our construction of the perturbed polynomial $p = f - q$ is similar to the one in [Sch05c], [NS07], or [Ave13] where the polynomial $h_{m,k}$ is also a univariate (sum of) squares. Their choice for $h_{k,m}$ is simpler, but it results in worst bounds for the degree and the norm of q, than the one we obtain using the plateau polynomial $h_{k,m}$. These univariate SoS coefficients have also been used in [KS15], to prove that one can uniformly approximate positive polynomials on compact sets, using the proper subcone of the quadratic module $\mathcal{Q}(\mathbf{g})$ where the SoS coefficient of g_i is of the form $\sum_j (h_j(g_i))^2$, for h_j univariate. They derive a Putinar's Positivstellensatz and apply it to Polynomial Optimization problems. We describe the equivalent of this uniform approximation result in Theorem 2.4.1.

We now turn our attention to the estimate of k and s. Recall that these parameters have to be chosen in such a way, for all $x \in D$

$$p(x) = f(x) - q(x) = f(x) - s\sum_{i=1}^{r} h_{k,m}\left(\frac{g_i(x)}{\|g_i\|}\right)^2 \frac{g_i(x)}{\|g_i\|} \geq \frac{f^*}{2} \tag{2.15}$$

see **B.1** in Section 2.2.2.

Proposition 2.2.12. *Assume that the normalization assumption* (2.1) *is satisfied. If the inequalities*

$$s > \frac{6\|f\|}{\delta}; \tag{2.16}$$

$$k > \frac{4(r-1)}{\delta} + 1; \tag{2.17}$$

$$k > \frac{8rs}{f^*}; \tag{2.18}$$

are satisfied, then (2.15) *is satisfied for all $x \in D$.*

Proof. Let $x \in A$ so that $G(x) \geq \delta$, i.e. $\min\{\frac{g_1(x)}{\|g_1\|}, \dots, \frac{g_r(x)}{\|g_r\|}\} \leq -\delta$ (see (2.3) and (2.6)), and without loss of generality assume $\frac{g_1(x)}{\|g_1\|} \leq -\delta$. Notice that from Proposition 2.2.11 we have $h_{k,m}(\frac{g_1(x)}{\|g_1\|})^2 \geq 1 - \frac{1}{k}$ and, if $\frac{g_1(x)}{\|g_1\|} \geq 0$, $h_{k,m}(\frac{g_1(x)}{\|g_1\|})^2 \leq \frac{2}{k}$. Then:

$$\begin{aligned}
p(x) &= f(x) - s\sum_{i=1}^{r} h_{k,m}\left(\frac{g_i(x)}{\|g_i\|}\right)^2 \frac{g_i(x)}{\|g_i\|} \\
&\geq f(x) + s\delta(1 - \frac{1}{k}) - s\sum_{i=1}^{r} h_{k,m}\left(\frac{g_i(x)}{\|g_i\|}\right)^2 \frac{g_i(x)}{\|g_i\|} \\
&\geq f(x) + s\delta(1 - \frac{1}{k}) - s\frac{2(r-1)}{k} = f(x) + s\frac{\delta}{2}(1 - \frac{1}{k}) + s(\frac{\delta}{2}(1 - \frac{1}{k}) - \frac{2(r-1)}{k}).
\end{aligned}$$

Assuming $k \geq 2$, from Equation (2.16) and Equation (2.17), we have respectively

$$f(x) + s\frac{\delta}{2}(1 - \frac{1}{k}) > -\|f\| + \frac{3}{2}\|f\| = \frac{\|f\|}{2} \geq \frac{f^*}{2}$$

and

$$\frac{\delta}{2}(1 - \frac{1}{k}) - \frac{2(r-1)}{k} > 0$$

, so that $p(x) > \frac{f^*}{2}$ for $x \in A$.

By Equation (2.18), $\frac{3f^*}{4} - \frac{2sr}{k} > \frac{f^*}{2}$. By the normalization assumptions (2.1) and as $h_{k,m}^2$ is upper bounded by $\frac{2}{k}$ on $[0,1]$ (see Proposition 2.2.11), we therefore deduce that for $x \in D \setminus A$:

$$p(x) = f(x) - s\sum_{i=1}^{r} h_{k,m}\left(\frac{g_i(x)}{\|g_i\|}\right)^2 \frac{g_i(x)}{\|g_i\|} \geq \frac{3f^*}{4} - sr\frac{2}{k} = \frac{3f^*}{4} - \frac{2sr}{k} > \frac{f^*}{2}$$

This shows that $p(x) > \frac{f^*}{2}$ for $x \in D = A \cup (D \setminus A)$. □

Proposition 2.2.11 and Proposition 2.2.12 determine constraints that the parameters k, s and m have to satisfy to obtain the bound **B.1** (or more explicitly (2.15)). With these constraints, we can finally obtain the bounds **B.2** and **B.3** as well.

Proposition 2.2.13. *Let p and q be as in* (2.15), *with* (2.16), (2.17), (2.18) *and the normalization assumptions* (2.1) *satisfied. Let $d(\mathbf{g}) = \max_i \deg g_i$. Then*

$$q \in \mathcal{Q}_{2md(\mathbf{g})+d(\mathbf{g})}(\mathbf{g}) \tag{2.19}$$

$$m = O(\mathfrak{c}^{\frac{4}{3}} r^{\frac{1}{3}} 2^{4\L} d^{\frac{8\L}{3}} \varepsilon(f)^{-\frac{4\L+1}{3}}). \tag{2.20}$$

$$\|p\| = O(\|f\| 2^{3\L} r\mathfrak{c} d(f)^{2\L} \varepsilon(f)^{-\L}), \tag{2.21}$$

$$\deg p = O(2^{4\L} r^{\frac{1}{3}} \mathfrak{c}^{\frac{4}{3}} d(\mathbf{g}) d(f)^{\frac{8\L}{3}} \varepsilon(f)^{-\frac{4\L+1}{3}}). \tag{2.22}$$

Proof. Equation (2.19) follows immediately from the definition of q, see eq. (2.12).

Now, let $d = d(f) = \deg f$. We proceed bounding m in terms of $\varepsilon(f)$.

We can choose $m = \left\lceil \frac{6}{\delta}\sqrt[3]{\frac{4(k-1)}{\pi}} + 3 \right\rceil$ from Proposition 2.2.11, thus it is enough to bound k and δ. From Proposition 2.2.9 we can choose $\delta = \frac{1}{\mathfrak{c}}(\frac{\varepsilon(f)}{8d^2})^{\L} = \mathfrak{c}^{-1}\varepsilon(f)^{\L} 2^{-3\L} d^{-2\L}$. From Equation (2.16) we deduce that:

$$s = O(\frac{\|f\|}{\delta}) = O(\|f\| \mathfrak{c} 2^{3\L} d^{2\L} \varepsilon(f)^{-\L}). \tag{2.23}$$

From Equation (2.17) we deduce that $k = O(\frac{r}{\delta})$, while from Equation (2.18) (together with Equation (2.16)) we deduce that $k = O(\frac{r}{\varepsilon(f)\delta})$: the latter has a higher order in terms of $\varepsilon(f)$, and thus finally:

$$k = O(\mathfrak{c} 2^{3\L} r d^{2\L} \varepsilon(f)^{-(\L+1)}). \tag{2.24}$$

Now we plug Equation (2.24) in $m = \left\lceil \frac{6}{\delta}\sqrt[3]{\frac{4(k-1)}{\pi}} + 3 \right\rceil$ and obtain:

$$m = O(\frac{k^{\frac{1}{3}}}{\delta}) = O((\mathfrak{c}^{\frac{1}{3}} r^{\frac{1}{3}} 2^{\L} d^{\frac{2\L}{3}} \varepsilon(f)^{-\frac{\L+1}{3}})(\mathfrak{c} 2^{3\L} d^{2\L} \varepsilon(f)^{-\L})) = O(\mathfrak{c}^{\frac{4}{3}} r^{\frac{1}{3}} 2^{4\L} d^{\frac{8\L}{3}} \varepsilon(f)^{-\frac{4\L+1}{3}}). \tag{2.25}$$

From the properties of $h_{k,m}$ (Proposition 2.2.11) and Equation (2.23) we obtain:

$$\begin{aligned}\|p\| &\leq \|f\| + s\sum_{i=1}^{r}\left\| h_{k,m}\left(\frac{g_i(x)}{\|g_i\|}\right)^2 \frac{g_i(x)}{\|g_i\|}\right\| \leq \|f\| + sr(1 + \frac{1}{k}) \\ &\leq \|f\| + 2sr = O(\|f\| + \|f\| \mathfrak{c} r 2^{3\L} d^{2\L} \varepsilon(f)^{-\L}) \\ &= O(\|f\| \mathfrak{c} r 2^{3\L} d^{2\L} \varepsilon(f)^{-\L}).\end{aligned}$$

Similarly, using Equation (2.25) we have:

$$\begin{aligned} \deg(p) = \deg(f-q) &\leq \max_i \{\deg(h_{k,m}(g_i/\|g_i\|)^2 g_i/\|g_i\|), i=1,\ldots,r\} \\ &= O(2md(\mathbf{g}) + d(\mathbf{g})) = O(2^{4Ł} r^{\frac{1}{3}} \mathfrak{c}^{\frac{4}{3}} d(\mathbf{g}) d^{\frac{8Ł}{3}} \varepsilon(f)^{-\frac{4Ł+1}{3}}), \end{aligned}$$

where $d(\mathbf{g}) = \max_i \deg g_i$. □

In this section, we have effectively constructed a polynomial Urysohn function $q \in \mathcal{Q}(\mathbf{g})$, with bounds **B.1**, **B.2** and **B.3** as in Section 2.2.2, in order to reduce the problem from the semialgebraic set S to the domain D. The key ingredient has been the construction of SoS coefficients $h_{k,m}(g_i/\|g_i\|)^2 \in \Sigma^2$ from uniform polynomial approximation. These SoS coefficients may be seen as a polynomial approximation of the Urysohn function $H(\frac{g_i(x)}{\|g_i\|})$, where $H(t)$ is the function introduced in (2.13).

It is important to remark that the cone Σ^2 with the grading given by the total degree, that is used to approximate continuous positive functions on D, can be replaced with any other hierarchy of cones of functions with similar approximation properties. In this way we obtain another approximation of $H(\frac{g_i(x)}{\|g_i\|})$. This approximation can be used to construct similarly a function q with bounds **B.1**, **B.2** and **B.3**. Indeed, the proofs above do not require to work with polynomials or sums of squares, but only Łojasiewicz inequalities and approximation properties of positive functions are needed. See also Section 2.6.2.

2.2.5 End of the proof

We can now conclude the proof of the main theorem, following the idea in Section 2.2.1.

Theorem 2.2.14. *Assume $n \geq 2$ and let $g_1, \ldots, g_r \in \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \ldots, x_n]$ satisfying the normalization assumption* (2.1). *Let $f \in \mathbb{R}[\mathbf{x}]$ such that $f^* = \min_{x \in S} f(x) > 0$. Let $\mathfrak{c}$, $Ł$ be the Łojasiewicz coefficient and exponent given by Definition 2.2.8. Then $f \in \mathcal{Q}_\ell(\mathbf{g})$ if*

$$\begin{aligned} \ell &\geq O(n^3 2^{5nŁ} r^n \mathfrak{c}^{2n} d(\mathbf{g})^n d(f)^{3.5nŁ} \varepsilon(f)^{-2.5nŁ}) \\ &= \gamma(n, \mathbf{g})\, d(f)^{3.5nŁ} \varepsilon(f)^{-2.5nŁ}, \end{aligned}$$

where $\gamma(n, \mathbf{g}) \geq 1$ depends only on n and $\mathbf{g}$.

In the proof, we choose $D = [-1,1]^n$ and we use an effective version of Schmüdgen's Positivstellensatz for the box $[-1,1]^n$.

Theorem 2.2.15 ([LS21]). *Let $f \in \mathbb{R}[\mathbf{x}]$, $\deg f = d$ and $f > 0$ on $[-1,1]^n$. Let $f_{\min} = \min_{x \in [-1,1]^n} f(x)$ and $f_{\max} = \max_{x \in [-1,1]^n} f(x)$. Then there exists a constant $C(n,d)$ (depending only on n and d) such that $f \in \mathcal{O}_{nr}(1 \pm x_i : i \in \{1, \ldots, n\})$, where:*

$$r \geq \max\left\{\pi d \sqrt{2n}, \sqrt{\frac{C(n,d)(f_{\max} - f_{\min})}{f_{\min}}}\right\}.$$

Moreover the constant $C(n,d)$ is a polynomial in d for fixed n:

$$C(n,d) \leq 2\pi^2 d^2 (d+1)^n n^3 = O(d^{n+2} n^3)$$

Our assumption is that $\mathcal{Q}(1-\|\mathbf{x}\|_2^2) \subset \mathcal{Q}(\mathbf{g})$, while Theorem 2.2.15 involves $\mathcal{O}(1 \pm x_i : i \in \{1,\dots,n\})$. But we have already shown in Lemma 2.2.1 that we can move from the latter to the former with a constant degree shift.

We are now ready to prove the main theorem.

Proof of Theorem 2.2.14. We choose $D = [-1,1]^n$. Let $p = f - q = f - s\sum_{i=1}^r h_{k,m}(g_i/\|g_i\|)^2 g_i/\|g_i\|$ be as in Equation (2.15), with s,k,m satisfying *Equation* (2.16), *Equation* (2.17), *Equation* (2.18) and $h_{k,m}$ as in Proposition 2.2.11. In particular:

- $p \geq \frac{f^*}{2}$ on $[-1,1]^n$ from Proposition 2.2.12;
- $\|p\| = O(2^{3Ł} r\, \mathfrak{c}\, d(f)^{2Ł} \|f\| \varepsilon(f)^{-Ł})$ from Equation (2.21);
- $\deg p = O(2^{4Ł} r^{\frac{1}{3}} \mathfrak{c}^{\frac{4}{3}} d(\mathbf{g}) d(f)^{\frac{8Ł}{3}} \varepsilon(f)^{-\frac{4Ł+1}{3}})$ from Equation (2.22).

We apply Theorem 2.2.15 to p: $p \in \mathcal{O}_{n\ell_0}(1 \pm x_i : i \in \{1,\dots,n\})$, if $\ell_0 \geq \sqrt{\frac{C(n,\deg p)(p_{\max}-p_{\min})}{p_{\min}}}$. Recall also from Theorem 2.2.15 that $C(n,m) = O(n^3 m^{n+2})$. We now deduce the asymptotic order of ℓ_0:

$$
\begin{aligned}
\sqrt{\frac{C(n,\deg p)(p_{\max}-p_{\min})}{p_{\min}}} =& O\Big(\sqrt{n^3 (\deg p)^{n+2} (\frac{2\|p\|}{f^*} + 1)}\Big) \\
=& O\Big(\sqrt{n^3 (2^{4Ł} r^{\frac{1}{3}} \mathfrak{c}^{\frac{4}{3}} d(\mathbf{g}) d(f)^{\frac{8Ł}{3}} \varepsilon(f)^{-\frac{4Ł+1}{3}})^{n+2} \frac{\|f\| 2^{3Ł} r \mathfrak{c} d(f)^{2Ł} \varepsilon(f)^{-Ł}}{f^*}}\Big) \\
=& O\Big((n^3 2^{(4n+11)Ł} r^{\frac{n+5}{3}} \mathfrak{c}^{\frac{4n+11}{3}} d(\mathbf{g})^{n+2} d(f)^{\frac{2(4n+11)Ł}{3}} \varepsilon(f)^{-\frac{(4Ł+1)n+11Ł+5}{3}})^{\frac{1}{2}}\Big) \\
=& O\Big(n^{\frac{3}{2}} 2^{\frac{(4n+11)Ł}{2}} r^{\frac{n+5}{6}} \mathfrak{c}^{\frac{4n+11}{6}} d(\mathbf{g})^{\frac{n+2}{2}} d(f)^{\frac{(4n+11)Ł}{3}} \varepsilon(f)^{-\frac{(4Ł+1)n+11Ł+5}{6}}\Big),
\end{aligned}
$$

so we can choose $\ell_0 = O(n^{\frac{3}{2}} 2^{\frac{(4n+11)Ł}{2}} r^{\frac{n+5}{6}} \mathfrak{c}^{\frac{4n+11}{6}} d(\mathbf{g})^{\frac{n+2}{2}} d(f)^{\frac{(4n+11)Ł}{3}} \varepsilon(f)^{-\frac{(4Ł+1)n+11Ł+5}{6}})$ and $p \in \mathcal{O}_{n\ell_0}(1 \pm x_i : i \in \{1,\dots,n\})$. Now, from Lemma 2.2.1 we have $\mathcal{O}_{n\ell_0}(1 \pm x_i : i \in \{1,\dots,n\}) \subset \mathcal{Q}_{n\ell_0+n}(1-\|\mathbf{x}\|_2^2)$. Moreover from the normalization assumption we have that $1-\|\mathbf{x}\|_2^2 \in \mathcal{Q}(\mathbf{g})$. In particular if $1-\|\mathbf{x}\|_2^2 \in \mathcal{Q}_{\ell_1}(\mathbf{g})$ and thus $\mathcal{Q}_{n\ell_0+n}(1-\|\mathbf{x}\|_2^2) \subset \mathcal{Q}_{n\ell_0+n+\ell_1}(\mathbf{g})$, i.e. choosing $\ell = nO(\ell_0) = O(n^{\frac{5}{2}} 2^{\frac{(4n+11)Ł}{2}} r^{\frac{n+5}{6}} \mathfrak{c}^{\frac{4n+11}{6}} d(\mathbf{g})^{\frac{n+2}{2}} d(f)^{\frac{(4n+11)Ł}{3}} \varepsilon(f)^{-\frac{(4Ł+1)n+11Ł+5}{6}})$ we have $p \in \mathcal{Q}_\ell(\mathbf{g})$. Finally notice that $f = (f-p)+p$ and

- $p \in \mathcal{Q}_\ell(\mathbf{g})$ from the discussion above;
- $q = f - p \in \mathcal{Q}_\ell(\mathbf{g})$ from Proposition 2.2.13, since from the estimate of m in Proposition 2.2.13 we need a degree $\leq \ell$ for the representation of q in $\mathcal{Q}(\mathbf{g})$.

Then $f = p + q \in \mathcal{Q}_\ell(\mathbf{g})$ with

$$
\ell = O(n^{\frac{5}{2}} 2^{\frac{(4n+11)Ł}{2}} r^{\frac{n+5}{6}} \mathfrak{c}^{\frac{4n+11}{6}} d(\mathbf{g})^{\frac{n+2}{2}} d(f)^{\frac{(4n+11)Ł}{3}} \varepsilon(f)^{-\frac{(4Ł+1)n+11Ł+5}{6}}). \tag{2.26}
$$

We simplify the exponents for readability. Recall that $Ł \geq 1$ and $\mathfrak{c} \geq 1$, and assume $n \geq 2$. Under these assumptions the inequalities $(4n+11)Ł \leq 10nŁ$, $n+5 \leq 6n$, $4n+11 \leq 10n$, $n+2 \leq 2n$ and $(4Ł+1)n + 11Ł + 5 \leq 14nŁ$ hold. Therefore we deduce that $f \in \mathcal{Q}_\ell(\mathbf{g})$ if

$$
\begin{aligned}
\ell &\geq O(n^3 2^{5nŁ} r^n \mathfrak{c}^{2n} d(\mathbf{g})^n d(f)^{3.5nŁ} \varepsilon(f)^{-2.5nŁ}) \\
&= \gamma(n,\mathbf{g}) d(f)^{3.5nŁ} \varepsilon(f)^{-2.5nŁ},
\end{aligned}
$$

where $\gamma(n,\mathbf{g}) = O(n^3 2^{5n\text{Ł}} r^n \mathfrak{c}^{2n} d(\mathbf{g})^n) \geq 1$. □

Remark. From Equation (2.26), we have a sharper bound than the one presented in Theorem 2.2.14. The exponents in Theorem 2.2.14 have been simplified for the sake of readability and are not optimal.

2.3 Łojasiewicz inequalities for regular semialgebraic sets

We have seen in the previous section that the Łojasiewicz exponent Ł and constant $\mathfrak{c}$ play a key role in the bound of the Effective Putinar's Positivstellensatz. In this section we study these parameters under generic regularity conditions that are common in optimization. In particular, we show that under regularity conditions, the Łojasiewicz exponent Ł is equal to one, and we have explicit estimates for $\mathfrak{c}$.

We proceed proving explicitly the Łojasiewicz inequalities for the Euclidean distance E and the semialgebraic distance G. The proof goes as follows:

(i) we work first *locally* at $z \in \partial S$ considering only the g_i such that $g_i(z) = 0$;

(ii) we consider the cone of $y \in D$ such that $E(y) = \mathrm{d}(y,S) = \|y-z\|_2$;

(iii) we estimate $\|y-z\|_2$ using the linear part of the active inequalities that are negative at y, see Proposition 2.3.6;

(iv) we extend from the linear part of the inequalities to the inequalities;

(v) we move from results local at $z \in \partial S$ to global results, showing that Ł = 1 and giving estimates for $\mathfrak{c}$ (see Theorem 2.3.9 and Theorem 2.3.13).

2.3.1 Minimizers of the distance function

In this section, we work on points (i), (ii) and (iii). We first present an example showing that we can expect Ł = 1 under regularity conditions.

Example 2.3.1. Consider the univariate polynomial $g(x) = \frac{\varepsilon^2 - x^2}{1-\varepsilon^2}$ and let $S = S(g) = [-\varepsilon, \varepsilon] \subset [-1,1]$. Now let $x \in [-1,1]$ and E, G be as in Definition 2.2.8. It is easy to show that:

$$E(x) \leq \frac{1-\varepsilon^2}{2\varepsilon} G(x).$$

Indeed, if for example $\varepsilon \leq x \leq 1$, we have $E(x) = x - \varepsilon$, $G(x) = \frac{x^2-\varepsilon^2}{1-\varepsilon^2} = \frac{(x+\varepsilon)(x-\varepsilon)}{1-\varepsilon^2}$ and $D(x) = \frac{1-\varepsilon^2}{x+\varepsilon} G(x) \leq \frac{1-\varepsilon^2}{2\varepsilon} G(x)$. This shows that we can choose Ł = 1 for all $\varepsilon > 0$.

On the other hand if $\varepsilon = 0$, i.e. $g(x) = -x^2$ and $S = \{0\}$, we have a singular equation. A simple computation shows that it is not possible to choose Ł = 1 in this case. The minimum Ł satisfying the inequality is Ł = 2.

We introduce a regularity condition needed to prove Ł = 1, generalizing Example 2.3.1. This is a standard condition in optimization (see [Ber99, sec. 3.3.1]), which implies the so-called Karush–Kuhn–Tucker (KKT) conditions [Ber99, prop. 3.3.1].

Definition 2.3.2. Let $x \in \mathcal{S}(\mathbf{g})$. The active constraints at x are the constraints $g_{i_1}, \dots, g_{i_m}$ such that $g_{i_j}(x) = 0$. We say that the *Constraint Qualification Condition (CQC)* holds at x if for all active constraints $g_{i_1}, \dots, g_{i_l}$ at x, the gradients $\nabla g_{i_1}(x), \dots, \nabla g_{i_m}(x)$ are linearly independent.

We start working locally. For $z \in S$ we denote

$$I = I(z) = \{\, i \in \{1, \dots, r\} \mid g_i(z) = 0 \,\}$$

the indices corresponding to the *active constraints* at z. For $y \in D$ and $z \in S$ such that $E(y) = \left\|y - z\right\|$ we denote:

- $\mathbf{g} = \mathbf{g}(y) = (g_1(y), \dots, g_r(y))$;
- $\mathbf{g}_I = \mathbf{g}_I(y) = (g_i(y) \colon i \in I)$;
- $J = J(z) = \mathrm{Jac}(\mathbf{g}_I)(z) = \left(\frac{\partial g_i}{\partial x_j}\right)_{i \in I,\ j \in \{1, \dots, n\}}$ the transposed Jacobian matrix at z, that is the matrix whose columns are the entries of the gradients $\boldsymbol{\nabla} g_i(z)$ of the active constraints at z;
- $\mathbf{N}_I = \mathbf{N}_I(z) = \mathrm{Gram}(\boldsymbol{\nabla} g_i(z) \colon i \in I) = J^t J$ the Gram matrix at z;
- to simplify the notations, hereafter we assume that $\|g\| = 1$.

Definition 2.3.3. We denote $\sigma_J(z) = \sigma_{\min}(J(z))$ be the smallest singular value $\sigma_{\min}(J(z))$ of $J(z)$.

As $\mathbf{N}_I = J^t J$, notice that $\left\|\mathbf{N}_I^{-1}\right\| = \sigma_{\min}(\mathbf{N}_I)^{-1} = \sigma_{\min}(J)^{-2} = \sigma_J(z)^{-2}$.

We show now how we can use $J = J(z)$ to describe the cone of points y such that $E(y) = \mathrm{d}(y, S) = \left\|y - z\right\|_2$.

Lemma 2.3.4. *Let $y \in \mathbb{R}^n \setminus \mathcal{S}(\mathbf{g})$, and let z be a point in $S = \mathcal{S}(\mathbf{g})$ minimizing the distance of y to S, that is $E(y) = \mathrm{d}(y, S) = \left\|y - z\right\|_2$. If $\{\, g_i \colon i \in I \,\}$ are the active constraints at z and the CQC hold, then there exist $\lambda_i \in \mathbb{R}_{\geq 0}$ such that:*

$$y - z = \sum_{i \in I} \lambda_i \boldsymbol{\nabla}(-g_i)(z) = -J\boldsymbol{\lambda}.$$

Proof. Fix $y \in \mathbb{R}^n$. Notice that $y - x = -\frac{\boldsymbol{\nabla}\left\|y - x\right\|_2^2}{2}$, where the gradient is taken w.r.t. x. Moreover $z \in S$ such that $\mathrm{d}(y, S) = \left\|y - z\right\|_2$ is a minimizer of the following Polynomial Optimization Problem:

$$\min_x \frac{\left\|y - x\right\|_2^2}{2} \colon g_i(x) \geq 0 \ \forall i \in \{1, \dots, r\}.$$

Since the CQC holds at z, we deduce from [Ber99, prop. 3.3.1] that the KKT conditions hold. In particular:

$$\frac{\boldsymbol{\nabla}\left\|y - z\right\|_2^2}{2} = \sum_{i \in I} \lambda_i \boldsymbol{\nabla} g_i(z)$$

For some $\lambda_i \in \mathbb{R}_{\geq 0}$. Therefore $y - z = -\frac{\boldsymbol{\nabla}\mathrm{d}(y,z)^2}{2} = \sum_{i \in I} \lambda_i \boldsymbol{\nabla}(-g_i)(z)$. □

Let $\boldsymbol{\lambda} = \boldsymbol{\lambda}(y) := (\lambda_i(y); i \in I)$ be the column vector in Lemma 2.3.4, so that $(y-z) = -J\boldsymbol{\lambda}$. Note that $\boldsymbol{\lambda}(y)$ depends linearly on $y-z$ and is given by the formula

$$\boldsymbol{\lambda}(y) = -\mathbf{N}_I^{-1} J^t (y-z).$$

Then, using Taylor's expansion at z and Lemma 2.3.4, we obtain:

$$\mathbf{g}_I = \mathbf{g}_I(y) = J^t(y-z) + \mathbf{h} = -\mathbf{N}_I \boldsymbol{\lambda} + \mathbf{h} \tag{2.27}$$

and the Mean-value form for the remainder in Taylor's theorem gives:

$$\|\mathbf{h}\|_2 \leq \mathfrak{c}_2 \left\|y - z\right\|_2^2, \tag{2.28}$$

where $\mathfrak{c}_2 = \mathfrak{c}_2(\mathbf{g}) = \max_{x \in D}\{\|\mathrm{Hess}(g_i)\|_2, i = 1, \ldots, r\}$ denotes an upper bound for the second derivative of $\mathbf{g}_I$ on D.

We keep working locally at $z \in S$, and in particular considering only the active constraints at z, whose indexes are denoted $I(z) \subset \{1, \ldots, r\}$. Notice that, if $y \in D \setminus S$ is close enough to $z \in \partial S$, then $g_i(y) \leq 0$ implies $g_i(z) = 0$: so only the active constraints at z and negative at y determine the value of $G(y)$ in a neighborhood of z. We introduce a notation to identify those indices:

$$I_- = I_-(y, z) = \{\, j \in I = I(z) \mid g_j(y) \leq 0 \,\}. \tag{2.29}$$

Moreover we introduce the function $\widetilde{G_-}(y) = \left(\sum_{j \in I_-} g_j(y)^2\right)^{\frac{1}{2}}$ as intermediate step between G and E. Indeed, it is easy to upper bound $\widetilde{G_-}(y)$ in terms of $G(y)$:

$$\widetilde{G_-}(y) = \Big(\sum_{j \in I_-} g_j(y)^2\Big)^{\frac{1}{2}} \leq \sqrt{|I_-|}\max_{j \in I_-}\left|g_j(y)\right| \leq \sqrt{n}\, G(y). \tag{2.30}$$

For the last inequality, we are using the fact that CQC at z implies $|I_-| \leq |I| \leq n$, and recall that we are assuming that $\|g_i\| = 1$ for all i. So we only need to find an upper bound for $E(y)$ in terms of $\widetilde{G_-}(y)$. In order to do that, let $\mathbf{g}_I(y) = \mathbf{g}_-(y) + \mathbf{g}_+(y)$, where:

- $\mathbf{g}_-(y) = (\min\{0, g_i(y) \colon i \in I\})$ and
- $\mathbf{g}_+(y) = (\max\{0, g_i(y) \colon i \in I\})$,

and notice that $\left\|\mathbf{g}_-(y)\right\|_2 = \widetilde{G_-}(y)$.

We proceed similarly to analyze the linear part of $\mathbf{g}_I$. In the sequel we denote

$$\boldsymbol{\gamma} = \boldsymbol{\gamma}(y) = J^t(y-z) = -\mathbf{N}_I \boldsymbol{\lambda}(y) = \mathbf{N}_I \boldsymbol{\lambda} \tag{2.31}$$

the linear part of $\mathbf{g}_I$.

The idea is to show first the inequality for the linear part $\boldsymbol{\gamma}(y)$, and then extend it to $\mathbf{g}_I$. In particular we want to relate the norm $\left\|y - z\right\|_2 = \langle y-z, y-z\rangle$, computed with respect to the euclidean scalar product, with the norm of $\boldsymbol{\gamma}(y)$ w. r. t. another inner product. Exploiting (2.31), one see that

$$\langle y-z, y-z\rangle = \langle \boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle_{\mathbf{N}_I} = \langle \boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle_{\mathbf{N}_I^{-1}} \tag{2.32}$$

where $\langle\cdot,\cdot\rangle_{\mathbf{N}_I}$ denotes the inner product induced by $\mathbf{N}_I$: $\langle \boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle_{\mathbf{N}_I} = \boldsymbol{\lambda}^t N_I \boldsymbol{\lambda}$. Notice that both $\mathbf{N}_I$ and $\mathbf{N}_I^{-1}$ define an inner product since they are positive definite.

As in the case of $\mathbf{g}_I$, let

$$\tilde{I}_- = \tilde{I}_-(y,z) = \{ i \in I(z) \mid \gamma_i(y) \leq 0 \} \tag{2.33}$$

and $\boldsymbol{\gamma}(y) = \boldsymbol{\gamma}_-(y) + \boldsymbol{\gamma}_+(y)$, where:

- $\boldsymbol{\gamma}_-(y) = (\min\{0, \boldsymbol{\gamma}_i(y)\} : i \in I)$ and
- $\boldsymbol{\gamma}_+(y) = (\max\{0, \boldsymbol{\gamma}_i(y)\} : i \in I)$.

Lemma 2.3.5. *With the notation above, we have:*

- $\langle \boldsymbol{\gamma}_-, \boldsymbol{\gamma} \rangle_{\mathbf{N}_I^{-1}} \geq 0$;
- $\langle \boldsymbol{\gamma}_+, \boldsymbol{\gamma} \rangle_{\mathbf{N}_I^{-1}} \leq 0$
- $\langle \boldsymbol{\gamma}_+, \boldsymbol{\gamma}_- \rangle_{\mathbf{N}_I^{-1}} \leq 0$

Proof. For the first inequality notice that $\langle \boldsymbol{\gamma}_-, \boldsymbol{\gamma} \rangle_{\mathbf{N}_I^{-1}} = -\boldsymbol{\gamma}_-^t \boldsymbol{\lambda} = -\sum_{i \in \tilde{I}_-} \gamma_i \lambda_i \geq 0$ because all λ_i are non-negative. A similar argument shows the second inequality. Finally $\langle \boldsymbol{\gamma}_+, \boldsymbol{\gamma}_- \rangle_{\mathbf{N}_I^{-1}} = \langle \boldsymbol{\gamma}_+, \boldsymbol{\gamma} \rangle_{\mathbf{N}_I^{-1}} - \langle \boldsymbol{\gamma}_+, \boldsymbol{\gamma}_+ \rangle_{\mathbf{N}_I^{-1}} \leq 0$ as claimed. □

The following observation, crucial for the sequel, shows that we can bound $\left\| y - z \right\|_2$ only in terms of the negative γ_i.

Proposition 2.3.6. *With the notation above, we have:*

$$\left\| y - z \right\|_2 \leq \frac{1}{\sigma_J(z)} \Big(\sum_{i \in \tilde{I}_-} \gamma_i^2(y) \Big)^{\frac{1}{2}} = \frac{1}{\sigma_J(z)} \left\| \gamma_- \right\|_2 \tag{2.34}$$

where $\sigma_J(z)$ is the smallest singular value of J (see Definition 2.3.3).

Proof. Note that Lemma 2.3.5 implies the proposition since it shows that

$$\langle \boldsymbol{\gamma}, \boldsymbol{\gamma} \rangle_{\mathbf{N}_I^{-1}} = \langle \boldsymbol{\gamma}_+, \boldsymbol{\gamma} \rangle_{\mathbf{N}_I^{-1}} + \langle \boldsymbol{\gamma}_-, \boldsymbol{\gamma}_+ \rangle_{\mathbf{N}_I^{-1}} + \langle \boldsymbol{\gamma}_-, \boldsymbol{\gamma}_- \rangle_{\mathbf{N}_I^{-1}} \leq \langle \boldsymbol{\gamma}_-, \boldsymbol{\gamma}_- \rangle_{\mathbf{N}_I^{-1}}$$

and this allows us to complete (2.32) to get (2.34):

$$\left\| y - z \right\| = \langle y - z, y - z \rangle \leq \langle \boldsymbol{\gamma}, \boldsymbol{\gamma} \rangle_{\mathbf{N}_I^{-1}} \leq \langle \boldsymbol{\gamma}_-, \boldsymbol{\gamma}_- \rangle_{\mathbf{N}_I^{-1}} \leq \frac{1}{\sigma_J(z)} \Big(\sum_{i \in \tilde{I}_-} \gamma_i^2(y) \Big)^{\frac{1}{2}} = \frac{1}{\sigma_J(z)} \left\| \gamma_- \right\|$$

□

2.3.2 Łojasiewicz distance inequality

We can now describe Łojasiewicz exponent and constant between E and G when constraint qualification condition (Definition 2.3.2) holds. Recall that we are assuming that $\|g_i\| = 1$ for all i for sake of simplicity.

Let $\sigma_J = \inf_{z \in \partial S} \sigma_J(z) = \inf_{z \in \partial S} \sigma_{\min}(J(z))$. Notice that $\sigma_J > 0$ as ∂S is compact and $\sigma_{\min}(J(z))$ is lower semicontinuous. Let $I = I(z)$ and let $I_- = I_-(y) = \{i \in I : \mathbf{g}_i(y) \leq 0\}$. Note that we do not have necessarily that $I_- = \tilde{I}_-$ (see Equation (2.29) and Equation (2.33)): the sign of $g_i(y)$ might be different from the sign of $\gamma_i(z)$.

We want to move from $\boldsymbol{\gamma}$ to $\mathbf{g}_I$. To do this, we determine how close are $\mathbf{g}_-$ and γ_-.

Lemma 2.3.7. *With the notation above, we have:*

$$\left| \|\mathbf{g}_-\| - \left\|\boldsymbol{\gamma}_-\right\| \right| \leq \mathfrak{c}_2 \left\|y - z\right\|^2.$$

Proof. Note that if $g_i(y)$ and $\gamma_i(y)$ are of different signs then their absolute values are bounded by $\left|g_i(y) - \gamma_i(y)\right|$. Therefore, by standard triangle inequality,

$$\left| \|\mathbf{g}_-\| - \left\|\boldsymbol{\gamma}_-\right\| \right| = \Big| \Big(\sum_{i \in I_-} g_i^2(y) \Big)^{1/2} - \Big(\sum_{i \in \tilde{I}_-} \gamma_i^2(y) \Big)^{1/2} \Big| \leq \Big(\sum_{i \in I} (g_i(y) - \gamma_i(y))^2 \Big)^{1/2} = \|\mathbf{h}\| \leq \mathfrak{c}_2 \left\|y - z\right\|^2,$$

where the latter inequality follows from (2.28). □

We first show the Łojasiewicz inequality with $Ł = 1$ locally at z.

Proposition 2.3.8. *If* $E(y) = \left\|y - z\right\| \leq \frac{\sigma_J}{2\mathfrak{c}_2}$ *then*

$$E(y) \leq \frac{2\sqrt{n}}{\sigma_J} G(y).$$

Proof. Fix $y \notin S$ such that $E(y) \leq \frac{\sigma_J}{2\mathfrak{c}_2}$ and $z \in \partial S$ such that $\left\|y - z\right\| = E(y)$. If $E(y) \leq \frac{\sigma_J}{2\mathfrak{c}_2}$ or, equivalently $\frac{\mathfrak{c}_2}{\sigma_J} E^2(y) \leq \frac{1}{2} E(y)$, then by Proposition 2.3.6 and Lemma 2.3.7 we have

$$\begin{aligned} E(y) = \left\|y - z\right\| &\leq \frac{1}{\sigma_J} \left\|\boldsymbol{\gamma}_-\right\| \leq \frac{1}{\sigma_J} \|\mathbf{g}_-\| + \frac{1}{\sigma_J} \mathfrak{c}_2 \left\|y - z\right\|^2 \\ &\leq \frac{1}{\sigma_J} \|\mathbf{g}_-\| + \frac{1}{2} E(y). \end{aligned}$$

This implies the claimed inequality as $\|\mathbf{g}_-\| = \widetilde{G}_-(y) \leq \sqrt{n}\, G(y)$ (since $|I_-(z,y)| \leq |I(z)| \leq n$ under CQC at z). □

We are finally able to prove that $Ł = 1$. We denote $U = \{y \in D \mid E(y) < \frac{\sigma_J}{2\mathfrak{c}_2}\}$ the open neighborhood of S of points at distance $< \frac{\sigma_J}{2\mathfrak{c}_2}$.

Theorem 2.3.9. *Suppose that the CQC holds at every point of* $\mathcal{S}(\mathbf{g})$*. Then, for all* $y \in D$,

$$E(y) \leq \mathfrak{c}_E G(y),$$

with $\mathfrak{c}_E = \sup\{\frac{E(y)}{G(y)} \mid y \in D \setminus S\} \leq \max(\frac{2\sqrt{n}}{\sigma_J}, \frac{\operatorname{diam}(D)}{G^*})$, *where*

$$G^* = \min_{y \in D \setminus U} G(z) > 0 \tag{2.35}$$

and $\operatorname{diam}(D) = \max_{x,y \in D} \left\|x - y\right\|$.

Proof. If $E(y) \leq \frac{\sigma_J}{2\mathfrak{c}_2}$ then by Proposition 2.3.8 we have

$$E(y) \leq \frac{2\sqrt{n}}{\sigma_J} G(y).$$

Otherwise:

$$E(y) = \|y - z\| \leq \operatorname{diam}(D) \leq \operatorname{diam}(D) \frac{G(y)}{G^*},$$

since $y, z \in D$ (notice that, as $G(x) > 0$ on the compact set $D \setminus U$, we have $G^* > 0$). □

We want now to give another description of the constant $\mathfrak{c}_E$ in Theorem 2.3.9 as distance from *singular systems*, following the approach of [Cuc+09]. In other words, we show how $\mathfrak{c}_E$ can be interpreted as the *condition number* of $\mathbf{g}$. See also [BC13] for more about condition numbers.

For $\mathbf{d} = (d_1, \dots, d_r)$, let $\mathbb{R}[\mathbf{x}]_{\mathbf{d}} := \mathbb{R}[\mathbf{x}]_{d_1} \times \cdots \times \mathbb{R}[\mathbf{x}]_{d_r}$ denote the systems of polynomials of bounded degree, which we equip with the Euclidean norm $\|\cdot\|$ with respect to the monomial basis in any component (another choice could be the apolar or Bombieri-Weil norm $\|\cdot\|_{d_i}$ in degree $\leq d_i$ in every component, see [Cuc+09]).

We say that a system $\mathbf{g}$ is singular if there exists a point in $x \in \mathbb{R}^n$ such that $x \in \mathcal{S}(\mathbf{g})$ and the active constraints have rank deficient Jacobian at x. In other words, this is the set of systems $\mathbf{g}$ such that CQC does not hold at some point of the semi-algebraic set S defined by $\mathbf{g}$. Formally:

$$\begin{aligned} \mathrm{Sing} := \Big\{ \mathbf{g} \in \mathbb{R}[\mathbf{x}]_{\mathbf{d}} \mid \exists x \in \mathbb{R}^n : \bigvee_{Z \subset \{1,\dots,r\}} \Big(& g_j(x) = 0 \quad \forall j \in Z \\ & \wedge g_j(x) > 0 \quad \forall j \notin Z \\ & \wedge \operatorname{rank} \operatorname{Jac}(g_j(x) : j \in Z) < \min(n, |Z|) \Big) \Big\} \end{aligned} \tag{2.36}$$

We want to relate the constant $\mathfrak{c}_E$ in Theorem 2.3.9 with $\mathrm{d}(\mathbf{g}, \mathrm{Sing})$, the distance from $\mathbf{g}$ to the singular systems induced from the Euclidean norm. Notice that Sing is a semi-algebraic set (by Tarski–Seidenberg principle [BCR98, th. 2.2.1] or quantifier elimination[BCR98, prop. 5.2.2]), and therefore $\mathrm{d}(\cdot, \mathrm{Sing})$ is a well-defined continuous semi-algebraic function [BCR98, prop. 2.2.8].

Lemma 2.3.10. *Under the normalization assumption* (2.1) *and with the previous notations, we have* $\mathrm{d}(\mathbf{g}, \mathrm{Sing}) \leq \sqrt{2}\sigma_J$.

Proof. Let $z \in \partial S$ be such that $\sigma_J = \sigma_{\min}(J(z))$. Since the CQC hold at z, $\operatorname{rank} J(z)$ is maximal. On the following, we assume that all the inequalities are active at z, the general case being a trivial generalization. By the Eckart-Young theorem, the distance of $J(z)$ from rank deficient matrices is equal to $\sigma_{\min}(J(z))$: there exists P (of rank one) such that $J(z) - P$ has not maximal rank and $\|P\|_F = \|P\|_2 = \sigma_{\min}(J(z))$. Now consider a system $\mathbf{l}$ of affine equations vanishing at z and such that $\operatorname{Jac}(\mathbf{l})(z) = P$. Therefore, $\mathbf{g} - \mathbf{l} \in \mathrm{Sing}$ since $\operatorname{Jac}(\mathbf{g} - \mathbf{l})(z) = J(z) - P$ is rank deficient and $(\mathbf{g} - \mathbf{l})(z) = 0$. Now, notice that:

$$\mathrm{d}(\mathbf{g}, \mathrm{Sing}) \leq \|\mathbf{g} - (\mathbf{g} - \mathbf{l})\|_2 = \|\mathbf{l}\|_2$$

Write $\mathbf{l} = l_1, \dots l_r$ and $l_i(x) = l_{i0} + \sum_{j=1}^n l_{ij} x_j$. By hypothesis $l_i(z) = 0$ and $\|z\|_2^2 \leq 1$ (from the normalization assumption). Therefore:

$$l_{i0}^2 = (\sum_{i=1}^n l_i x_i)^2 \leq \|(l_{i1}, \dots, l_{in})\|_2^2 \|z\|_2^2 \leq \sum_{j=1}^n l_{ij}^2$$

Notice also that $\sigma_J^2 = \|P\|_F^2 = \sum_{i=1}^r \sum_{j=1}^n l_{ij}^2$, and thus:

$$\mathrm{d}(\mathbf{g}, \mathrm{Sing})^2 \leq \|\mathbf{l}\|_2^2 = \sum_{i=1}^r \sum_{j=1}^n l_{ij}^2 + \sum_{i=1}^r l_{i0}^2 \leq 2 \sum_{i=1}^r \sum_{j=1}^n l_{ij}^2 = 2\sigma_J^2$$

which concludes the proof. □

In order to measure the distance to Sing, we introduce a global equivalent to G^* in theorem 2.3.9. We define then:

$$\widetilde{G}^* := \min_{y \in \mathbb{R}^n \setminus U} G(z) > 0 \tag{2.37}$$

.

Lemma 2.3.11. *Let U be as in Theorem 2.3.9 and assume that $\widetilde{G}^* = G(y)$ is not attained on ∂U. Then $\frac{1}{\widetilde{G}^*} \leq \sqrt{r}\,\mathrm{d}(\mathbf{g}, \mathrm{Sing})^{-1}$.*

Proof. Recall that we are assuming $\|g_i\| = 1$ for all i and without loss of generality assume that $g_1(y) = -\widetilde{G}^*$. Since $y \notin \partial U$ we have $\nabla g_1(y) = 0$. Then the system $(g_1 + \widetilde{G}^*, \ldots, g_r + \widetilde{G}^*) \in \mathrm{Sing}$ is a singular system, and $\left\|\mathbf{g} - (g_1 + \widetilde{G}^*, \ldots, g_r + \widetilde{G}^*)\right\|_2 = \sqrt{r}\,\widetilde{G}^*$. Therefore $\mathrm{d}(\mathbf{g}, \mathrm{Sing}) \leq \sqrt{r}\,\widetilde{G}^*$, and finally $\frac{1}{\widetilde{G}^*} \leq \frac{\sqrt{r}}{\mathrm{d}(\mathbf{g},\mathrm{Sing})}$. □

Lemma 2.3.12. *Assume that $\widetilde{G}^* = G(y)$ is attained at $y \in \partial\{y \in D \mid E(y) \leq \frac{\sigma_J}{2\mathfrak{c}_2}\}$. Then $\frac{1}{\widetilde{G}^*} \leq \frac{4\sqrt{n}\mathfrak{c}_2}{\sigma_J^2}$.*

Proof. Since $E(y) = \frac{\sigma_J}{2\mathfrak{c}_2}$, we can apply Proposition 2.3.8:

$$\frac{\sigma_J}{2\mathfrak{c}_2} = E(y) \leq 2\sigma_J^{-1}\|\mathbf{g}_-\| \leq 2\sqrt{n}\,\sigma_J^{-1} G(y) = 2\sqrt{n}\,\sigma_J^{-1}\widetilde{G}^*.$$

Therefore $\frac{1}{\widetilde{G}^*} \leq 4\sqrt{n}\mathfrak{c}_2\sigma_J^{-2}$. □

We deduce from these two lemmas the following bound on Łojasiewicz constant in terms of the distance from $\mathbf{g}$ to the singular locus:

Theorem 2.3.13. *Suppose that the CQC holds at every point of $\mathcal{S}(\mathbf{g})$ and that the normalization assumption (2.1) is satisfied. Then, for all $y \in D$,*

$$E(y) \leq \max\Big(\frac{\mathfrak{c}}{\mathrm{d}(\mathbf{g}, \mathrm{Sing})}, \frac{8\,\mathrm{diam}(D)\sqrt{n}\,\mathfrak{c}_2}{\mathrm{d}(\mathbf{g}, \mathrm{Sing})^2}\Big)G(y),$$

where $\mathfrak{c}_2 = \mathfrak{c}_2(\mathbf{g}) = \max_{x \in D}\{\|\mathrm{Hess}(g_i(x))\|_2, i = 1, \ldots, r\}$ and $\mathfrak{c}_1 = \max(2\sqrt{2n}, \mathrm{diam}(D)\sqrt{r})$.

Proof. We estimate the constant $\mathfrak{c}_E = \sup\{\frac{E(y)}{G(y)} \mid y \in D \setminus S\} \leq \max(\frac{2\sqrt{n}}{\sigma_J}, \frac{\mathrm{diam}(D)}{G^*})$ in Theorem 2.3.9 using the previous lemmas. In particular, from Lemma 2.3.10 we have $\frac{1}{\sigma_J} \leq \frac{\sqrt{2}}{\mathrm{d}(\mathbf{g},\mathrm{Sing})}$, and using Lemma 2.3.11 and Lemma 2.3.12 we obtain:

$$\begin{aligned}
\frac{2\sqrt{n}}{\sigma_J} &\leq \frac{2\sqrt{2n}}{\mathrm{d}(\mathbf{g}, \mathrm{Sing})} \\
\frac{\mathrm{diam}(D)}{G^*} &\leq \frac{\mathrm{diam}(D)}{\widetilde{G}^*} \leq \mathrm{diam}(D)\max(\frac{4\sqrt{n}\mathfrak{c}_2}{\sigma_J^2}, \frac{\sqrt{r}}{\mathrm{d}(\mathbf{g}, \mathrm{Sing})}) \\
&\leq \mathrm{diam}(D)\max(\frac{8\sqrt{n}\mathfrak{c}_2}{\mathrm{d}(\mathbf{g}, \mathrm{Sing})^2}, \frac{\sqrt{r}}{\mathrm{d}(\mathbf{g}, \mathrm{Sing})})
\end{aligned}$$

Choosing $\mathfrak{c}_1 = \max(2\sqrt{2n}, \mathrm{diam}(D)\sqrt{r})$ we then see that $\mathfrak{c}_E \leq \max\Big(\frac{\mathfrak{c}}{\mathrm{d}(\mathbf{g},\mathrm{Sing})}, \frac{8\,\mathrm{diam}(D)\sqrt{n}\,\mathfrak{c}_2}{\mathrm{d}(\mathbf{g},\mathrm{Sing})^2}\Big)$, concluding the proof. □

Remark. Under a regularity condition, we have analyzed in Theorem 2.3.9 and Theorem 2.3.13 the Łojasiewicz constant, giving estimates for it, and moreover shown that the Łojasiewicz exponent is equal to one. On the contrary when the problem is not regular the bounds on the exponent $Ł$ can be large. We have:

$$Ł \leq d(\mathbf{g})(6d(\mathbf{g})-3)^{n+r}$$

see [KS15, sec. 3.1] and [KSS16] and the errata [KSS19].

We now present the corollary of Theorem 2.2.14 in regular cases, when we have show that the Łojasiewicz exponent is equal to one, and we have estimates for the Łojasiewicz constant.

Corollary 2.3.14. *Assume $n \geq 2$ and let $g_1,\dots,g_r \in \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1,\dots,x_n]$ satisfying the normalization assumption* (2.1) *and such that the CQC (Definition 2.3.2) hold at every point of $\mathcal{S}(\mathbf{g})$. Let $f \in \mathbb{R}[\mathbf{x}]$ such that $f^* = \min_{x\in S} f(x) > 0$. Then $f \in \mathcal{Q}_\ell(\mathbf{g})$ if*

$$\ell = O(n^3 2^{5n} r^n \mathfrak{c}_E^{2n} d(\mathbf{g})^n d(f)^{3.5n} \varepsilon(f)^{-2.5n}),$$

where $\mathfrak{c}_E$ is given by Theorem 2.3.9 or Theorem 2.3.13.

Proof. Apply Theorem 2.2.14, Theorem 2.3.9 and Theorem 2.3.13. □

2.4 Convergence of Lasserre's relaxations optimum

In this section, we apply the Effective Putinar's Positivstellensatz (Theorem 2.2.14) to compute convergence rates of Lasserre's hierarchies. In particular our goal is to prove Theorem 2.4.2, where we show for the first time a polynomial convergence for Lasserre's hierarchies: this is possible since the Effective Putinar's Positivstellensatz has a polynomial and not exponential dependence on $\varepsilon(f)^{-1}$.

We briefly recall the basics of Polynomial Optimization, referring to Section 1.6 for more details. Let $f, g_1,\dots,g_r \in \mathbb{R}[\mathbf{x}]$. The goal of Polynomial Optimization is to find:

$$f^* = \inf \left\{ f(x) \in \mathbb{R} \mid x \in \mathbb{R}^n,\ g_i(x) \geq 0 \ \text{ for } i = 1,\dots,r \right\} = \inf_{x\in\mathcal{S}(\mathbf{g})} f(x),$$

that is the infimum f^* of the *objective function* f on the *basic closed semialgebraic set* $S = \mathcal{S}(\mathbf{g})$.

The *SoS relaxation of order* ℓ of the problem above is $\mathcal{Q}_{2\ell}(\mathbf{g})$ and the supremum:

$$f^*_{\mathrm{SoS},\ell} = \sup \left\{ \lambda \in \mathbb{R} \mid f - \lambda \in \mathcal{Q}_{2\ell}(\mathbf{g}) \right\}.$$

The *moment relaxation of order* ℓ of the problem above is $\mathcal{L}_{2\ell}(\mathbf{g}) = \mathcal{Q}_{2\ell}(\mathbf{g})^\vee$ and the infimum:

$$f^*_{\mathrm{Mom},\ell} = \inf \left\{ \langle \Lambda | f \rangle \in \mathbb{R} \mid \Lambda \in \mathcal{L}^{(1)}_{2\ell}(\mathbf{g}) \right\}.$$

Recall that $f^*_{\mathrm{SoS},\ell} \leq f^*_{\mathrm{Mom},\ell} \leq f^*$ for all ℓ. Thus, the convergence results of this section, stated for the SoS hierarchy $(f^*_{\mathrm{SoS},\ell})_{\ell\in\mathbb{N}}$, are also valid for the moment hierarchy $(f^*_{\mathrm{Mom},\ell})_{\ell\in\mathbb{N}}$.

A first step for the proof of Theorem 2.4.3 is to recognize Theorem 2.2.14 as a quantitative result of approximation of positive polynomials on $\mathcal{S}(\mathbf{g})$ with polynomials in the truncated quadratic module.

Theorem 2.4.1. *Assume $n \geq 2$ and let $\mathbf{g}$ satisfy the normalization condition* (2.1). *Let $Ł$ be the Łojasiewicz exponent defined in Definition 2.2.8 and let $f \geq 0$ on $\mathcal{S}(\mathbf{g})$. Then for $0 < \varepsilon \leq \|f\|$, we have $f - f^* + \varepsilon = q \in \mathcal{Q}_\ell(\mathbf{g})$ for*

$$\ell \geq \gamma'(n,\mathbf{g})\, d(f)^{3.5nŁ}\, \|f\|^{2.5nŁ} \varepsilon^{-2.5nŁ} \tag{2.38}$$

where $\gamma'(n,\mathbf{g}) = 3^{2.5nŁ}\gamma(n,\mathbf{g}) \geq 1$ depends only on n and $\mathbf{g}$ and $\gamma(n,\mathbf{g})$ is given by Theorem 2.2.14.

Proof. Notice that $f - f^* + \varepsilon > 0$ on $\mathcal{S}(\mathbf{g})$ and

$$\varepsilon(f - f^* + \varepsilon) = \frac{\varepsilon}{\|f - f^* + \varepsilon\|} \geq \frac{\varepsilon}{\|f\| + |f^*| + \varepsilon} \geq \frac{\varepsilon}{3\|f\|}$$

for $\varepsilon \leq \|f\|$. Moreover $\deg f - f^* + \varepsilon = \deg f = d(f)$. By Theorem 2.2.14, we have $f - f^* + \varepsilon = q \in Q_\ell(\mathbf{g})$ if

$$\begin{aligned}\ell &\geq O(n^3 2^{5nŁ} r^n \mathfrak{c}^{2n} d(\mathbf{g})^n d(f)^{3.5nŁ} (\frac{\varepsilon}{3\|f\|})^{-2.5nŁ}) \\ &= \gamma'(n,\mathbf{g})\, d(f)^{3.5nŁ}\, \|f\|^{2.5nŁ} \varepsilon^{-2.5nŁ}\end{aligned}$$

where $\gamma'(n,\mathbf{g}) = 3^{2.5nŁ}\gamma(n,\mathbf{g}) = O(n^3 2^{5nŁ} 3^{2.5nŁ} r^n \mathfrak{c}^{2n} d(\mathbf{g})^n) \geq 1$ depends only on n and $\mathbf{g}$, and not on f, and $\gamma(n,\mathbf{g})$ is given by Theorem 2.2.14. ☐

Remark. From Equation (2.26), we have $\gamma(n,\mathbf{g}) = O(n^{\frac{3}{2}} 2^{\frac{4Łn+11Ł}{2}} r^{\frac{n+5}{6}} \mathfrak{c}^{\frac{4n+11}{6}} d(\mathbf{g})^{\frac{n+2}{2}})$, where $\mathfrak{c}, Ł$ are defined in Definition 2.2.8. The exponents of $\gamma'(n,\mathbf{g}) = 3^{2.5nŁ}\gamma(n,\mathbf{g})$ in the proof have been simplified for the sake of readability and are not optimal.

Remark. Theorem 2.4.1 is a quantitative version of Weierstrass approximation theorem for positive polynomials on S, showing that a polynomial $f \in \mathrm{Pos}(S(\mathbf{g}))$ can be approximated uniformly on S (within distance ε) by an element $f^* + q \in Q_\ell(\mathbf{g})$ for $\ell \geq \gamma'(n,\mathbf{g})\, d(f)^{3.5nŁ}\, \|f\|^{2.5nŁ} \varepsilon^{-2.5nŁ}$.

We are now ready to prove the rate of convergence for Lasserre's hierarchies.

Theorem 2.4.2. *With the same hypothesis of Theorem 2.4.1, let $f^*_{\mathrm{SoS},\ell}$ be the Lasserre SoS relaxation. Then $f^* - f^*_{\mathrm{SoS},\ell} \leq \varepsilon$ for*

$$\ell \geq \gamma'(n,\mathbf{g})\, d(f)^{3.5nŁ}\, \|f\|^{2.5nŁ} \varepsilon^{-2.5nŁ}. \tag{2.39}$$

Proof. Notice that

$$f^*_{\mathrm{SoS},\ell} = \sup\{\lambda \in \mathbb{R} \mid f - \lambda \in \mathcal{Q}_{2\ell}(\mathbf{g})\} = \inf\{\varepsilon \in \mathbb{R}_{\geq 0} \mid f - f^* + \varepsilon \in \mathcal{Q}_{2\ell}(\mathbf{g})\}.$$

By Theorem 2.4.1, for $\ell \geq \gamma'(n,\mathbf{g})\, d(f)^{3.5nŁ}\, \|f\|^{2.5nŁ} \varepsilon^{-2.5nŁ}$, $f - f^* + \varepsilon \in \mathcal{Q}_\ell(\mathbf{g})$. This implies that $f^* - f^*_{\mathrm{SoS},\ell} \leq \varepsilon$ and concludes the proof. ☐

Theorem 2.4.3. *With the same hypothesis of Theorem 2.4.2 and $\gamma''(n,\mathbf{g}) = \gamma'(n,\mathbf{g})^{\frac{1}{2.5nŁ}}$, we have*

$$0 \leq f^* - f^*_{\mathrm{SoS},\ell} \leq \gamma''(n,\mathbf{g}) \|f\| d(f)^{\frac{7}{5}} \ell^{-\frac{1}{2.5nŁ}}.$$

Proof. We apply Theorem 2.4.2 with $\varepsilon \leq \|f\|$ such that $\ell = \lceil \gamma'(n,\mathbf{g}) d(f)^{3.5nŁ} \|f\|^{2.5nŁ} \varepsilon^{-2.5nŁ} \rceil$ and $\gamma''(n,\mathbf{g}) = \gamma'(n,\mathbf{g})^{\frac{1}{2.5nŁ}}$. ☐

In conclusion Theorem 2.2.14 allows proving Theorem 2.4.3, which shows a convergence of the Lasserre's lower approximations to f^*, polynomial in ℓ. This is an improvement in comparison with [NS07, th. 8], where the convergence is logarithmic in order ℓ of the hierarchy.

In regular polynomial optimization problems we can simplify the bound, since $Ł = 1$ in this case (see Section 2.3.2).

Corollary 2.4.4. *With the same hypothesis of Theorem 2.4.2 and $\gamma''(n,\mathbf{g}) = \gamma'(n,\mathbf{g})^{\frac{1}{2.5n}}$, we have*

$$0 \leq f^* - f^*_{\mathrm{SoS},\ell} \leq \gamma''(n,\mathbf{g})\|f\|d(f)^{\frac{7}{5}}\ell^{-\frac{1}{2.5n}}$$

if the CQC (Definition 2.3.2) hold at every point of $\mathcal{S}(\mathbf{g})$.

Proof. Apply Theorem 2.4.3 and Theorem 2.3.9 or Theorem 2.3.13. □

2.5 Convergence of pseudo-moment sequences to measures

In Section 2.4 we studied convergence of the *optima* $f^*_{\mathrm{SoS},d}$ and $f^*_{\mathrm{Mom},d}$. Moreover, we have seen in Theorem 2.4.1 that the Effective Putinar's Positivstellensatz can be used to study the convergence of the feasible set $\mathcal{Q}_{2d}(\mathbf{g})$ of the SoS relaxations to $\mathrm{Pos}(\mathcal{S}(\mathbf{g}))$. On the dual side, one natural question is then still open: how good is the approximation of the measures $\mathcal{M}(\mathcal{S}(\mathbf{g}))$ using the feasible set of the moment relaxation, namely the truncated positive linear functionals $\mathcal{L}_\ell(\mathbf{g}) = \mathcal{Q}_\ell(\mathbf{g})^\vee$? In particular, we focus on the sections $\mathcal{L}_d^{(1)}(\mathbf{g})$ and $\mathcal{M}^{(1)}(\mathcal{S}(\mathbf{g}))$.

To be able to compare relaxations of different order, in the following we often restrict the linear functionals to polynomials of degree $\leq t$, that is we consider the cones:

$$\mathcal{L}_\ell(\mathbf{g})^{[t]} = \{\Lambda^{[t]} \in \mathbb{R}[\mathbf{x}]_t^* \mid \Lambda \in \mathcal{L}_\ell(\mathbf{g})\}$$

where for $t \leq \ell$ we denote $\Lambda^{[t]}$ the restriction of $\Lambda \in \mathbb{R}[\mathbf{x}]_\ell^*$ to $\mathbb{R}[\mathbf{x}]_t^*$. See Section 1.3.8 and Section 1.6.2 for more details.

Recall in particular that, if $\mu \in \mathcal{M}(S)^{[t]} = \{\Lambda_\mu^{[t]} \mid \mu \in \mathcal{M}(S)\}$ and $q \in \mathcal{Q}_\ell(\mathbf{g}) \cap \mathbb{R}[\mathbf{x}]_t$ then $\langle \mu | q \rangle = \int q \,\mathrm{d}\mu \geq 0$, since $q \geq 0$ on S. In other words: $\mathcal{M}(S)^{[t]} \subset \mathcal{L}_\ell(\mathbf{g})^{[t]}$ for all ℓ, i.e. our dual cone is an outer approximation of the cone of measures supported on S. To compare quantitatively these cones we consider their affine sections $\mathcal{M}^{(1)}(S)^{[t]}$ and $\mathcal{L}_\ell^{(1)}(\mathbf{g})^{[t]}$. Recall that $\mathcal{L}_\ell^{(1)}(\mathbf{g})^{[t]}$ is a generating section of $\mathcal{L}_\ell(\mathbf{g})^{[t]}$ when $t \leq \frac{\ell}{2}$, see Lemma 1.3.9.

The main result of this section is Theorem 2.5.9, which shows the convergence of the outer approximation as ℓ goes to infinity, and deduce this convergence rate from Theorem 2.2.14. To measure this convergence we use the Hausdorff distance of sets $\mathrm{d_H}(\cdot,\cdot)$.

Before the proof of the main theorem, recall that in the finite dimensional vector space $\mathbb{R}[\mathbf{x}]_t$, all the norms are equivalent: we specify in Lemma 2.5.1 a constant that we will need in the proof of Theorem 2.5.7, for the following norms. For $f = \sum_{|\alpha|\leq t} a_\alpha \mathbf{x}^\alpha \in \mathbb{R}[\mathbf{x}]_t$, as usual $\|f\| = \max_{\mathbf{x}\in[-1,1]^n} |f(\mathbf{x})|$, and $\|f\|_2 = \sqrt{\sum_{|\alpha|\leq t} a_\alpha^2}$.

Lemma 2.5.1. *For $f \in \mathbb{R}[\mathbf{x}]_t$, we have $\|f\| \leq \sqrt{\binom{n+t}{t}}\|f\|_2$.*

Proof. Let $x \in [-1,1]^n$ such that $|f(x)| = \|f\|$, where $f = \sum_{|\alpha|\le t} a_\alpha \mathbf{x}^\alpha$. Denote $\mathbf{b}_t = (\mathbf{x}^\alpha)_{|\alpha|\le t}$ the monomial basis and $\mathrm{vec}(f) = (a_\alpha)_{|\alpha|\le t}$. Then:

$$\|f\| = |f(x)| = |\mathbf{b}_t(x)\cdot \mathrm{vec}(f)| \le \|\mathrm{vec}(f)\|_2 \|\mathbf{b}_t(x)\|_2 = \|f\|_2 \|\mathbf{b}_t(x)\|_2$$

using the Cauchy-Schwarz inequality. Finally, notice that $|x^\alpha| \le 1$ for all α since $x \in [-1,1]^n$, and thus $\|\mathbf{b}_t(x)\|_2 \le \sqrt{\dim \mathbb{R}[\mathbf{x}]_t} = \sqrt{\binom{n+t}{t}}$, which implies $\|f\| \le \sqrt{\binom{n+t}{t}}\|f\|_2$. □

We recall a version of Haviland's or Richter-Tchakaloff's theorem, that characterize linear functionals represented by measures supported on a compact set. See also [Sch17, th.17.3] and [Lau09, th. 5.13].

Theorem 2.5.2 ([Tch57])**.** *Let $S \subset \mathbb{R}^n$ be compact and let $\mathrm{Pos}(S)_t = \{f \in \mathbb{R}[\mathbf{x}] \mid \deg f \le t,\ f(x) \ge 0\ \forall x \in S\}$. Then for a linear functional $\Lambda \in \mathbb{R}[\mathbf{x}]_t^*$, $\Lambda \in \mathcal{M}(S)^{[t]}$ if and only if $\langle \Lambda | f\rangle \ge 0$ for all $f \in \mathrm{Pos}(S)_t$. In other words, $\mathcal{M}(S)^{[t]} = (\mathrm{Pos}(S)_t)^\vee$.*

We slightly modify Theorem 2.5.2 in order to consider only polynomials of unit norm.

Corollary 2.5.3. *Let $P = \{f \in \mathrm{Pos}(S)_t \mid \|f\|_2 = 1\}$ and let $\Lambda \in \mathbb{R}[\mathbf{x}]_t^*$. Then $\Lambda \in \mathcal{M}(S)^{[t]} \subset \mathbb{R}[\mathbf{x}]_t^*$ if and only if $\langle \Lambda | f\rangle \ge 0$ for all $f \in P$.*

Proof. Notice that $\langle \Lambda | f\rangle \ge 0 \iff \left\langle \Lambda \middle| \frac{f}{\|f\|_2} \right\rangle \ge 0$. Then apply Theorem 2.5.2. □

We interpret Corollary 2.5.3 in terms of convex geometry. The convex set

$$\mathcal{M}(S)^{[t]} = \{\, \Lambda \in \mathbb{R}[\mathbf{x}]_t^* \mid \forall f \in P, \langle \Lambda | f\rangle \ge 0 \,\}$$

is the convex cone dual to P. Any $f \in P$ is defining a hyperplane $\langle \Lambda | f\rangle = 0$ in $\mathbb{R}[\mathbf{x}]_t^*$, and an associated halfspace $H_f = \{\Lambda \in \mathbb{R}[\mathbf{x}]_t^* \mid \langle \Lambda | f\rangle \ge 0\}$ such that $\mathcal{M}(S)^{[t]} \subset H_f$. Corollary 2.5.3 means that $\mathcal{M}(S)^{[t]} = \bigcap_{f\in P} H_f$.

We consider a relaxation of the positivity condition to prove our convergence.

Definition 2.5.4. For $\varepsilon \ge 0$ and P as in Corollary 2.5.3, we define $C(\varepsilon) = \{\Lambda \in \mathbb{R}[\mathbf{x}]_t^* \mid \forall f \in P, \langle \Lambda | f\rangle \ge -\varepsilon\}$.

Notice that by definition and Corollary 2.5.3 we have $C(0) = \mathcal{M}(S)^{[t]}$. We show now that $C(\varepsilon)$ contains the truncated positive linear functionals of total mass one for a large enough order of the hierarchy.

Lemma 2.5.5. *Let $\ell \ge \gamma'(n,\mathbf{g})\, t^{3.5nL} \binom{n+t}{t}^{\frac{5nL}{4}} \varepsilon^{-2.5nL}$ and $t \le \ell/2$, where $\mathbf{g}$ satisfy the normalization assumption* (2.1) *and $\gamma'(n,\mathbf{g})$ is given by Equation* (2.38)*. Then $\mathcal{L}_\ell^{(1)}(\mathbf{g})^{[t]} \subset C(\varepsilon)$.*

Proof. By Lemma 2.5.1, for all $f \in P$ we have $\|f\| \le \binom{n+t}{t}^{\frac{1}{2}}$. From Theorem 2.4.1, we deduce that for $\ell \ge \gamma'(n,\mathbf{g})\, t^{3.5nL} \binom{n+t}{t}^{\frac{5nL}{4}} \varepsilon^{-2.5nL}$, we have $f - f^* + \varepsilon = q \in \mathcal{Q}_\ell(\mathbf{g})$. Thus for $\Lambda \in \mathcal{L}_\ell^{(1)}(\mathbf{g})^{[t]}$ we obtain $\langle \Lambda | f + \varepsilon\rangle = \langle \Lambda | q + f^*\rangle \ge 0$. Therefore, $\langle \Lambda | f\rangle \ge -\varepsilon$: this shows that $\mathcal{L}_\ell^{(1)}(\mathbf{g})^{[t]} \subset C(\varepsilon)$. □

The convex set $C(\varepsilon)$ can be seen as a *tubular* neighborhood of $\mathcal{M}(S)^{[t]}$. We are going to bound its Hausdorff distance to the measures (recall that the Hausdorff distance $\mathrm{d_H}(\cdot,\cdot)$ is defined as $\mathrm{d_H}(A,B) := \max\left\{\sup_{a\in A} \mathrm{d}(a,B),\ \sup_{b\in B} \mathrm{d}(A,b)\right\}$). We state and prove the result in the general setting of convex geometry, and finally use it to prove Theorem 2.5.7.

Lemma 2.5.6. *Let $C = \bigcap_{H\in\mathcal{H}} H$ be a closed convex set described as intersection of half spaces $H = \{\mathbf{x} \in \mathbb{R}^N \mid c_H \cdot \mathbf{x} + b_H \geq 0\}$, where*

- $\|c_H\|_2 = 1$ *for all $H \in \mathcal{H}$;*
- *$\mathcal{H}$ is the set of all the half-spaces containing C (of unit normal).*

If $H(\varepsilon) = \{\mathbf{x} \in \mathbb{R}^N \mid c_H \cdot \mathbf{x} + b_H \geq -\varepsilon\}$ and $C(\varepsilon) = \bigcap_{H\in\mathcal{H}} H(\varepsilon)$, then $\mathrm{d_H}(C, C(\varepsilon)) \leq \varepsilon$.

Proof. By definition $C \subset C(\varepsilon)$. Assume that this inclusion is proper, otherwise there is nothing to prove, and let $\xi \in C(\varepsilon) \setminus C$. Consider the closest point η in C of ξ on C, and the half space $H = \{\mathbf{x} \in \mathbb{R}^N \mid \frac{\eta-\xi}{\|\eta-\xi\|_2} \cdot \mathbf{x} + b \geq 0\} \in \mathcal{H}$ defined by the affine supporting hyperplane orthogonal to $\eta - \xi$ passing through η (and thus $\frac{\eta-\xi}{\|\eta-\xi\|_2} \cdot \eta = -b$). Notice that $H \in \mathcal{H}$ since H is defined by a normalized supporting hyperplane of C.

Finally notice that $\left\|\eta - \xi\right\|_2 = \frac{(\eta-\xi)\cdot(\eta-\xi)}{\|\eta-\xi\|_2} = -\frac{\eta-\xi}{\|\eta-\xi\|_2} \cdot \xi + \frac{\eta-\xi}{\|\eta-\xi\|_2} \cdot \eta = -(\frac{\eta-\xi}{\|\eta-\xi\|_2} \cdot \xi + b)$. Since $\xi \in C(\varepsilon)$ and $H \in \mathcal{H}$, we have $(\frac{\eta-\xi}{\|\eta-\xi\|_2} \cdot \xi + b) \geq -\varepsilon$, and thus $0 < \left\|\eta - \xi\right\|_2 \leq \varepsilon$. Then the distance between any $\xi \in C(\varepsilon) \setminus C$ and its closest point $\eta \in C$ is $\leq \varepsilon$, which implies $\mathrm{d_H}(C, C(\varepsilon)) \leq \varepsilon$. □

We are now ready to prove the first important result of the section.

Theorem 2.5.7. *Let $\mathcal{Q}(\mathbf{g})$ be a quadratic module where $\mathbf{g}$ satisfy the normalization assumption* (2.1) *and let*

$$\ell \geq \gamma'(n, \mathbf{g})\, t^{3.5nL} \binom{n+t}{t}^{\frac{5nL}{4}} \varepsilon^{-2.5nL}$$

with $\gamma'(n, \mathbf{g})$ given by Equation (2.38). *Then $\mathrm{d_H}(\mathcal{M}(S)^{[t]}, \mathcal{L}_\ell^{(1)}(\mathbf{g})^{[t]}) \leq \varepsilon$.*

Proof. By Corollary 2.5.3 we have:

$$\mathcal{M}(S)^{[t]} = \{\Lambda \in \mathbb{R}[\mathbf{x}]_t^* \mid \forall f \in P, \langle \Lambda | f \rangle \geq 0\} = \cap_{f\in P} H_f,$$

where $H_f = \{\Lambda \in \mathbb{R}[\mathbf{x}]_t^* \mid \langle \Lambda | f \rangle \geq 0\}$ with $\|f\|_2 = 1$ and $f \in \mathrm{Pos}(S)_t$. We check that the hyperplanes H_f with $f \in P$ defining $\mathcal{M}(S)^{[t]}$ satisfy the hypothesis of Lemma 2.5.6:

- The half-space H_f has a unit normal since $\|f\|_2 = 1$;
- Any supporting hyperplane of $\mathcal{M}(S)^{[t]}$ defines a halfspace $H_f = \{\Lambda \in \mathbb{R}[\mathbf{x}]_t^* \mid \langle \Lambda | f \rangle \geq 0\}$ with $f \in P$. Indeed if f defines a supporting hyperplane of $\mathcal{M}(S)^{[t]}$, then $\langle \mu | f \rangle = \int f \,\mathrm{d}\mu \geq 0$ for all $\mu \in \mathcal{M}(S)^{[t]}$. In particular for all $x \in S$ we have $f(x) = \int f \,\mathrm{d}\delta_x \geq 0$ (where δ_x denotes the Dirac measure concentrated at x). This proves that $f \in \mathrm{Pos}(S)_t$ and, normalizing it, we can assume $f \in P$.

Then from Lemma 2.5.6 we have $\mathrm{d_H}(\mathcal{M}(S)^{[t]}, C(\varepsilon)) \leq \varepsilon$.

Finally by Lemma 2.5.5 we deduce that $\mathcal{L}_\ell^{(1)}(\mathbf{g})^{[t]} \subset C(\varepsilon)$ and conclude that

$$\mathrm{d_H}(\mathcal{M}(S)^{[t]}, \mathcal{L}_\ell^{(1)}(\mathbf{g})^{[t]}) \leq \mathrm{d_H}(\mathcal{M}(S)^{[t]}, C(\varepsilon)) \leq \varepsilon.$$

□

Notice that in Theorem 2.5.7 we are bounding the distance between normalized linear functionals and measures that may be *not* normalized (i.e. not a probability measure). In the next section we solve this problem.

2.5.1 Convergence to probability measures

We recall and adapt to our context [JH16, lem. 3] to obtain a bound on the norm of pseudo-moment sequences. In particular we do not assume that the ball constraint is an explicit inequality, but only that the quadratic module is Archimedean.

Lemma 2.5.8. *Assume that $r^2 - \|\mathbf{x}\|_2^2 = q \in \mathcal{Q}_{\ell_0}(\mathbf{g})$. Then for all $t \in \mathbb{N}$ and $\ell \geq 2t - 2 + \ell_0$, if $\Lambda \in \mathcal{L}_\ell^{(1)}(\mathbf{g})$ we have $\left\|\Lambda^{[2t]}\right\|_2 \leq \sqrt{\binom{n+t}{t}\sum_{k=0}^t r^{2k}}$.*

Proof. For $\Lambda \in \mathcal{L}_\ell^{(1)}(\mathbf{g})$, let H_Λ^k be the Moment matrix of Λ in degree $\leq 2k$, which is positive semidefinite. Let $\left\|H_\Lambda^k\right\|_F$ be its Frobenius norm, i.e. $\left\|H_\Lambda^k\right\|_F = \sqrt{\sum_{|\alpha|,|\beta|\leq k} \Lambda_{\alpha+\beta}^2}$ (equal also to the square root of the sum of the singular values), and $\left\|H_\Lambda^k\right\|_2$ its ℓ^2 operator norm, i.e. the maximal eigenvalue of H_Λ^k (equal to its largest singular value). Notice that by definition we have $\left\|\Lambda^{[2k]}\right\|_2 \leq \left\|H_\Lambda^k\right\|_F$ and $\left\|H_\Lambda^k\right\|_2 \leq \sqrt{\operatorname{tr} H_\Lambda^k}$, Moreover recall $\left\|H_\Lambda^k\right\|_F \leq \sqrt{\operatorname{rank}(H_\Lambda^k)}\left\|H_\Lambda^k\right\|_2$. To obtain a bound on $\left\|\Lambda^{[2k]}\right\|_2$, we are going to use $\operatorname{tr} H_\Lambda^k = \sum_{|\alpha|\leq k} \Lambda_{2\alpha} = \left\langle \Lambda^{[2k]} \middle| \sum_{|\alpha|\leq k} \mathbf{x}^{2\alpha}\right\rangle$. As for $k \leq t$,

$$(r^2 - \|\mathbf{x}\|_2^2)(\sum_{|\alpha|\leq k-1} \mathbf{x}^{2\alpha}) \in \mathcal{Q}_{2t-2+\ell_0}(\mathbf{g}) \subset \mathcal{Q}_\ell(\mathbf{g})$$

we have

$$\begin{aligned} 0 &\leq \left\langle \Lambda \middle| (r^2 - \|\mathbf{x}\|_2^2)(\sum_{|\alpha|\leq k-1} \mathbf{x}^{2\alpha})\right\rangle = r^2 \left\langle \Lambda \middle| \sum_{|\alpha|\leq k-1} \mathbf{x}^{2\alpha}\right\rangle - \left\langle \Lambda \middle| \|\mathbf{x}\|_2^2 (\sum_{|\alpha|\leq k-1} \mathbf{x}^{2\alpha})\right\rangle \\ &= r^2 \operatorname{tr} H_\Lambda^{k-1} - \left(\left\langle \Lambda \middle| \sum_{|\alpha|\leq k} \mathbf{x}^{2\alpha}\right\rangle - \langle \Lambda | 1 \rangle\right) = r^2 \operatorname{tr} H_\Lambda^{k-1} + 1 - \operatorname{tr} H_\Lambda^k, \end{aligned}$$

that is, $\operatorname{tr} H_\Lambda^k \leq r^2 \operatorname{tr} H_\Lambda^{k-1} + 1$. Since $\operatorname{tr} H_\Lambda^0 = \Lambda_0 = 1$, we deduce by induction on k that $\operatorname{tr} H_\Lambda^t \leq \sum_{k=0}^t r^{2k}$ and thus

$$\left\|\Lambda^{[2t]}\right\|_2 \leq \left\|H_\Lambda^t\right\|_F \leq \sqrt{\operatorname{rank}(H_\Lambda^t)}\left\|H_\Lambda^t\right\|_2 \leq \sqrt{\binom{n+t}{t}\operatorname{tr} H_\Lambda^t} \leq \sqrt{\binom{n+t}{t}\sum_{k=0}^t r^{2k}}.$$

□

Finally we are ready to prove Theorem 2.5.9, where we obtain the bound for the distance between normalized linear functionals and probability measures.

Theorem 2.5.9. *Assume $n \geq 2$ and that the normalization assumptions* (2.1) *are satisfied, and in particular that $1 - \|\mathbf{x}\|_2^2 = q \in \mathcal{Q}_{\ell_0}(\mathbf{g})$. Let $0 < \varepsilon \leq \frac{1}{2}$, $t \in \mathbb{N}_+$ and $\ell \in \mathbb{N}$ such that $\ell \geq \gamma(n,\mathbf{g})\, 6^{2.5n\L}\, t^{6n\L} \binom{n+t}{t}^{2.5n\L} \varepsilon^{-2.5n\L}$ and $\ell \geq 2t + \ell_0$, with $\gamma(n,\mathbf{g})$ given by Theorem 2.2.14. Then*

$$\mathrm{d_H}(\mathcal{M}^{(1)}(S)^{[2t]}, \mathcal{L}_\ell^{(1)}(\mathbf{g})^{[2t]}) \leq \varepsilon.$$

Proof. Let $\varepsilon' = \frac{1}{2}\varepsilon t^{-1}\binom{n+t}{t}^{-\frac{1}{2}} \leq \frac{1}{4}$, $\Lambda \in \mathcal{L}_\ell^{(1)}(\mathbf{g})^{[2t]}$ and $\mu \in \mathcal{M}(S)^{[2t]}$ be the closest point to Λ. We first bound the norm of μ. As

$$\ell \geq \gamma(n,\mathbf{g})\, 6^{2.5nE}\, t^{6nE} \binom{n+t}{t}^{\frac{5nE}{2}} \varepsilon^{-2.5nE} = \gamma'(n,\mathbf{g})\, t^{3.5nE} \binom{n+t}{t}^{\frac{5nE}{4}} (\varepsilon')^{-2.5nE},$$

by Theorem 2.5.7 we have $\mathrm{d}(\Lambda,\mu) \leq \varepsilon'$.

Let $\mu_0 = \int 1\,\mathrm{d}\mu$. We want to bound the distance between Λ and $\frac{\mu}{\mu_0} \in \mathcal{M}^{(1)}(S)^{[2t]}$. Notice that

$$\mathrm{d}(\Lambda, \frac{\mu}{\mu_0}) \leq \mathrm{d}(\Lambda,\mu) + \mathrm{d}(\mu, \frac{\mu}{\mu_0}) \leq \varepsilon' + \left|\frac{1-\mu_0}{\mu_0}\right| \left\|\mu\right\|_2. \tag{2.40}$$

Since $\Lambda_0 = 1$, $\mathrm{d}(\Lambda,\mu) \leq \varepsilon'$ implies $1-\varepsilon' \leq \mu_0 \leq 1+\varepsilon'$, and therefore $\left|\frac{1-\mu_0}{\mu_0}\right| \leq \frac{\varepsilon'}{1-\varepsilon'}$. Moreover, using Lemma 2.5.8 we have

$$\left\|\mu\right\|_2 = \left\|\mu - \Lambda + \Lambda\right\| \leq \mathrm{d}(\mu,\Lambda) + \|\Lambda\|_2 \leq \varepsilon' + (t+1)\sqrt{\binom{n+t}{t}}.$$

Then from Equation (2.40) we conclude that

$$\mathrm{d}(\Lambda, \frac{\mu}{\mu_0}) \leq \varepsilon' + \frac{\varepsilon'}{1-\varepsilon'}(\varepsilon' + (t+1)\sqrt{\binom{n+t}{t}}) = \frac{\varepsilon'}{1-\varepsilon'} + \frac{\varepsilon'}{1-\varepsilon'}(t+1)\sqrt{\binom{n+t}{t}} \leq 2\varepsilon' t\sqrt{\binom{n+t}{t}} = \varepsilon,$$

since $\varepsilon' \leq \frac{1}{4}$, $n \geq 1$ and $t \geq 1$. □

As usual, we provide the better bound available in regular cases.

Corollary 2.5.10. *With the hypothesis of Theorem 2.5.9 and the CQC (Definition 2.3.2) satisfied at every point of $\mathcal{S}(\mathbf{g})$, then*

$$\mathrm{d_H}(\mathcal{M}^{(1)}(S)^{[2t]}, \mathcal{L}_\ell^{(1)}(\mathbf{g})^{[2t]}) \leq \varepsilon$$

if $\ell \geq \gamma(n,\mathbf{g})\, 6^{2.5n}\, t^{6n} \binom{n+t}{t}^{2.5nE} \varepsilon^{-2.5n}$.

Proof. Apply Theorem 2.5.9 and Theorem 2.3.9 or Theorem 2.3.13. □

In Theorem 2.5.9 we prove a bound for the convergence of Lasserre truncated pseudo-moments to moments of measures. The convergence, without bounds, can be deduced from [Sch05c, th. 3.4] by taking as objective function a constant. On the other hand, we can deduce [Sch05c, th. 3.4] from Theorem 2.5.9, by considering the sections of $\mathcal{L}_\ell^{(1)}(\mathbf{g})^{[t]}$ given by $\langle\Lambda|f\rangle = f^*_{\mathrm{Mom},k}$.

2.6 Perspectives

We have shown the first polynomial bound on the Effective Putinar's Positivstellensatz, and then applied it to deduce convergence rates for optima of Lasserre's hierarchies. Moreover, we have described bounds for the approximation of positive polynomials with elements of quadratic modules, and on the dual side we described bounds for the approximation of measures using truncated positive linear functionals.

From the results obtained and their proofs, several open questions naturally arise.

2.6.1 Bound improvements

In the proof of the Effective Putinar's Positivstellensatz, we have chosen $D = [-1,1]^n$ (see Section 2.2.5), and then applied Theorem 2.2.15. The main issue in our bound is the presence of n in the exponent of $\varepsilon(f)^{-1}$: this implies bad bounds when the number of variables is big. The n in the exponent is present due to the constant $C(n,d) = O(d^{n+2}n^3)$ in Theorem 2.2.15, that has an n as exponent. We plan to remove this dependence on n in the follow-up work [BMP22], where we replace:

- $D = [-1,1]^n$ with the n-dimensional simplex $D = \Delta_n$;
- Theorem 2.2.15 with an effective version of Polya's theorem [PR01].

The convergence rate for Polya's theorem is worse than the one in Theorem 2.2.15, but the constant is better as there is no n in the exponent. However, some care is needed since the norm used in [PR01] is the maximum absolute value of the coefficients with respect to the Bernstein basis, rather the max norm on D that we used in this chapter to deduce the Effective Putinar's Positivstellensatz.

2.6.2 Generalized Lasserre's hierarchies

Through the chapter, we used SoS polynomials for the proof of the Effective Putinar's Positivstellensatz, but we may have worked with more general cones. Indeed, the important features that we used are two:

(i) the approximation of positive continuous functions on D, to construct q such that $p = f - q \geq \frac{f^*}{2}$ on D (that is, the construction of $h_{k,m}(g_i/\|g_i\|)^2$);

(ii) the representation of strictly positive functions on D (in particular, of p) as an element of a certain cone (that is, $p \in \mathcal{O}(1 \pm x_i : i \in \{1,\dots,n\})$).

This observation leads immediately to more general statements, replacing the SoS cone with different ones.

For instance, consider C_d the (linear) cone $C_d = \sum_{|\alpha_i|\leq d} \mathbb{R}_{\geq 0} \cdot B_{d,\alpha}(\mathbf{x})$, where $B_{d,\alpha}(\mathbf{x})$ are the elements of the degree d Bernstein basis. In other words, C_d is the standard positive orthant with respect to the Bernstein basis. From the well-known approximation properties of the Bernstein basis, the two points above are satisfied. Thus, one can consider the cone:

$$\mathbf{C}_d(\mathbf{g}) = C_d + C_{d-\deg g_1} g_1 + \cdots + C_{d-\deg g_r} g_r$$

and the proofs in this chapter show that:

$$\overline{\bigcup_{d\in\mathbb{N}} \mathbf{C}_d(\mathbf{g})} = \mathrm{Pos}(\mathcal{S}(\mathbf{g})),$$

i.e. we have a representation of strictly positive polynomials, as in the case of Putinar's Positivstellensatz. We can therefore design generalized Lasserre's hierarchies using these cones and their duals. Moreover, using convergence rates for Bernstein approximation, we can also get degree bound for the representation and convergence rates for the hierarchies, as in SoS case.

Notice that this construction gives a hierarchy of *linear* inner approximations $\mathbf{C}_d(\mathbf{g})$ of $\text{Pos}(\mathcal{S}(\mathbf{g}))$. Although we expect this approximation to have less *exact* representation properties than the SoS one (see Chapter 3), the use of linear cones might allow going to higher relaxation order, getting better lower approximations of f^*. This construction with the Bernstein basis is related with Handelman's theorem ([Han88]), that have been already considered in Polynomial Optimization, see e.g. [Las15, th. 2.23] and the following discussion.

Another natural extension is to replace polynomials with more general functions on a given domain: for instance, one could consider cones of piecewise polynomial functions (splines), that have good approximation properties, and build generalized Lasserre's hierarchies from them.

2.6.3 Generalized moment problems

In the context of Generalized Moment Problems (GMP), general convergence to moments of measures has been studied in [Tac21]. The uniform bounded mass assumption in [Tac21] is trivially satisfied in the context of Polynomial Optimization, since $\Lambda_0 = \langle \Lambda | 1 \rangle = 1$: the convergence result of [Tac21] is thus more general than [Sch05c, th. 3.4] and the one implied by Theorem 2.5.9. But we conjecture, and leave it for future exploration, that it is possible to extend the proof technique of Theorem 2.5.9 to the GMP and give bounds on the rate of convergence also in this extended context.

2.6.4 Certificates of emptiness

In all our discussion, we assumed that the semialgebraic set $\mathcal{S}(\mathbf{g})$ is nonempty. However, it would be interesting to adapt our argument to the empty case, to give degree bounds for the certificates of emptiness of $\mathcal{S}(\mathbf{g})$: $-1 = s_0 + s_1 g_1 + \dots s_r g_r \in \mathcal{Q}(\mathbf{g})$ (in the Archimedean case). Of special interest would be the case of rational inequalities $g_i \in \mathbb{Q}[\mathbf{x}]$. In this case, we could relate the bounds with the magnitude of the coefficients of the polynomials, and obtain a bound involving the bitsize.

Giving degree bound for such a problem is hard. For instance, in [LPR20] the authors compute elementary recursive degree bounds for Kivrine-Stengle Positivstellensatz, in particular certifying emptiness of (general) semialgebraic sets. They obtain as bound a tower of five exponentials. To be able to compare our estimates with [LPR20], we need to consider the worst case bounds for the Łojasiewicz exponent (see [KS15, sec. 3.1], [KSS16] and the errata [KSS19]) and for the Łojasiewicz constant.

Chapter 3

Exactness and Flat Truncation in Polynomial Optimization

This chapter is based on [BM22a].

3.1 Context and results

In polynomial optimization, Lasserre's hierarchies are used to compute lower approximations of the minimum f^* of the objective function f on the basic closed semialgebraic set $S(\mathbf{g})$ defined by the tuple of polynomials $\mathbf{g} = g_1, \dots g_r$, see Section 1.6. In practice *finite convergence* of these lower approximations often happens, i.e. the lower approximation given by the relaxation at some order d coincide with f^*. Even if this practical behavior is observed and the sum of squares hierarchy can naturally provide *certificates of positivity* (with a representation in the quadratic module), we cannot verify finite convergence of the hierarchies using this representation. Moreover, it is not possible to extract the minimizers only from the sum of squares hierarchy.

But positive answers to these problems are provided from the moment hierarchy. Indeed, it is well-know that using the flat extension criterion introduced by Curto and Fialkow (see Section 1.4.1) we can certify the finite convergence of Lasserre's moment hierarchy, and extract the minimizers (see e.g. [Lau09; Las15]). If this criterion is satisfied, the truncated pseudo-moment sequence realizing the minimum coincides with the sequence of moments of a measure supported at the minimizers: in other words, we have a representation of the positive linear functional defined by the pseudo-moment sequence as a measure acting on polynomials.

The flat truncation criterion in thus widely used in polynomial optimization, but not completely understood theoretically. This chapter is devoted to the study of this criterion, together with the related concept of *exactness*. We also study how these properties of the moment hierarchy are (and are not) related with exactness and finite convergence properties of the sum of squares hierarchy, namely with certificates of positivity on the sum of squares side.

We first introduce the concept of *exactness* for the moment hierarchy, in order to study the outer approximation of the measures supported at the minimizers of f on S at any order of the relaxation. Then, we highlight its connections with the flat truncation property, used to certify finite convergence of the moment hierarchy. The main result of the chapter is the following theorem, giving the first necessary and sufficient criterion for flat truncation for the moment hierarchy of lower approximations of the problem: $\min f(x)\colon x \in \mathcal{S}(\mathbf{g})$.

Theorem 3.5.4. *Assume that we have moment finite convergence. Then* $\dim \frac{\mathbb{R}[\mathbf{x}]}{\mathrm{supp}(Q+(f-f^*))} = 0$ *if and only if there exists d such that a generic $\Lambda^* \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ has flat truncation.*

In particular, if $\rho = \rho(S^{\min})$, $D = \max(d_{\mathbf{g}}, \lceil \frac{\deg(f)}{2} \rceil)$ and $\delta \in \mathbb{N}$ is such that $f - f^* \in \overline{\mathcal{Q}_{2\delta}(\mathbf{g})}$, flat truncation happens for $\Lambda^* \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ generic at degree $\rho - 1$ when d is such that:

(i) $(\sqrt[\mathbb{R}]{\mathrm{supp}\, Q(\mathbf{g})})_{2\delta+2\rho+2D-\deg(f)-2} \subset \overline{\mathcal{Q}_{2d}(\mathbf{g})}$;

(ii) $\mathcal{I}(S^{\min})_{2\rho+2D-2} \subset \overline{\mathcal{Q}_{2d}(\mathbf{g}) + (f - f^*)_{2d}}$;

(iii) $\delta + 2\rho + 2D - \deg(f) - 2 \le d$.

In the theorem, $\mathcal{L}_{2d}^{\min}(\mathbf{g})$ denotes the feasible pseudo-moment sequences of the order d moment relaxation that realize the minimum, $d_{\mathbf{g}} := \lceil \frac{1}{2} \max_{i=1,\dots,s} \deg(g_i) \rceil$ (see Definition 3.4.17) and ρ is the Castelnuovo-Mumford regularity of the set of minimizers, see Section 3.4.3.

As already mentioned, even if extensively used the flat truncation (or flat extension) property was not completely understood theoretically in polynomial optimization (see Section 3.1.1). In the theorem, we provide the first necessary and sufficient condition for the flat truncation property, and give the first degree bounds for the order of the relaxation needed to achieve flat truncation.

The proof of the theorem requires a detailed analysis of the dual cones of (truncated) quadratic modules, and we show that the moment hierarchy coincides with an extended sum of squares hierarchy in Theorem 3.4.3 and Theorem 3.4.11. In particular, in Theorem 3.4.3 we characterize an extended quadratic module (that is the most natural to study the moment hierarchy) as the Minkowski sum of the original quadratic module and the real radical of its support. In Theorem 3.4.11 we conclude this analysis, showing that the annihilator of a truncated positive linear functional (or, in other words, the kernel of the associated moment matrix) generates the real radical of the support in degree high enough. These results are also motivated from the duality theory in conic programming, see Section 1.6.4.

The analysis required for the proof of the theorem gives a detailed understanding of the duality between Lasserre's moment and sum of squares hierarchies. It allows also to create several new examples and counterexamples for finite convergence and exactness properties of the sum of squares and moment hierarchies. For instance, we describe an optimization problem on a finite semialgebraic set with finite converge of the hierarchies, but whose convergence cannot be certified using flat truncation (see Example 3.3.12). See Examples 3.3.8, 3.3.9, 3.3.10, 3.3.11, for more.

A key ingredient for the proof of Theorem 3.5.4 is the analysis of the zero dimensional case (more precisely, the case when the support of the quadratic module is zero dimensional). Theorem 3.4.19 and Theorem 3.4.20 give a complete description of the correspondence between zero dimensionality and flat truncation, and provide degree bounds for the flat truncation property to hold. This generalize existing results for finite real varieties and preoderings defining zero dimensional semialgebraic sets, see Section 3.1.1.

As consequences of Theorem 3.5.4, we show that flat truncation holds under generic regularity properties (Theorem 3.5.7), and apply the result to finite semialgebraic sets (Theorem 3.5.11) and polar ideals (Theorem 3.5.15).

We present examples of the application of these results, and in particular study some instances of the Optimal Power Flow (OPF) Problem in Section 3.6.

3.1.1 Related works

Several works have been developed over the last decades to address SoS representation problems. [Par02] showed that if the complex variety $\mathcal{V}_{\mathbb{C}}(I)$ defined by an ideal I generated by real polynomials is finite and I is radical, then $f - f^*$ has a representation as a sum of squares modulo I. [Lau07] showed the finite convergence property if the complex variety $\mathcal{V}_{\mathbb{C}}(I)$ is finite, and a moment sequence representation property, if moreover the ideal I is radical. [Nie13c] showed that if the semialgebraic set S is finite, then the finite convergence property holds for a preordering defining S. [Sch05a] proved that $f - f^*$ is in the quadratic module $\mathcal{Q}$ defining S modulo $(f - f^*)^2$ if and only if $f - f^* \in \mathcal{Q}$ and then the SoS hierarchy is exact. [Mar06] and [Mar09] proved that, under regularity conditions at the minimizers, known as Boundary Hessian Conditions (BHC), $f - f^*$ is in the quadratic module and the SoS exactness property holds. [NDS06], [DNP07] showed that, by adding gradient constraints when $S = \mathbb{R}^n$ or KKT constraints when S is a general basic semialgebraic set, the SoS exactness property holds when the corresponding Jacobian ideal is radical. [Nie13a] showed that, by adding the Jacobian constraints, the finite convergence property holds under some regularity assumption. [Nie13b] showed that finite convergence and the flat truncation property are equivalent under generic assumptions, if the SoS hierarchy is exact and strong duality holds. In [Nie14], it is shown that BHC imply finite convergence and that BHC are generic. [KS19] showed the SoS exactness property if the quadratic module defining S is Archimedean and some strict concavity properties of f at the finite minimizers are satisfied.

On the side of sum of squares, our contributions are limited but interesting: Theorem 3.4.3 clarifies the relationship between *radicality assumptions* and the *closedness* of truncated quadratic modules: if the support of the quadratic module is not radical, we expect the existence of f such that $f - f^* \in \overline{\mathcal{Q}(\mathbf{g})_{2d}} \setminus \mathcal{Q}(\mathbf{g})_{2d}$ for some d. Then, if we minimize f on $S(\mathbf{g})$, we have $f^*_{\mathrm{SoS},d} = f^*$, but there is no certificate of positivity for $f - f^*$. See for instance Example 1.3.4.

On the other hand, the moment relaxations have been much less studied. We recall [LLR08] and [Las+13], which prove that if S is finite and defined by equalities (or by a preordering), the value f^*, the minimizers and the vanishing ideal of S can be recovered from moment matrices associated with a moment relaxation of finite order. We unify the description of the zero dimensional case for equations and preorderings in Theorem 3.4.19, Theorem 3.4.20 and Theorem 3.5.11. Our results are also more general than those of [LLR08; Las+13], see for instance Example 3.4.21.

In [Nie13b] it is shown that finite convergence and the flat truncation property are equivalent if the SoS hierarchy is exact and strong duality holds, and under further generic assumptions. These are strong hypothesis, and the result exploits properties of the SoS hierarchy rather than the moment one. For instance, they are not satisfied in the basic example Example 1.3.4. Moreover, it is not clear if these hypotheses are satisfied under the genericity assumptions needed for finite convergence result in [Nie14].

In this chapter we focus more on the moment hierarchy, and we provide solutions for the questions above. Theorem 3.5.4 gives the first necessary and sufficient conditions for flat truncation, and the key idea is to use *closures* of truncated quadratic modules (this allows to ignore radicality issues). This result gives also degree bounds on the order of the moment relaxation needed for the flat truncation property to hold, answering an open question in [Nie13b]. Moreover, we use this result to show that, under the generic assumptions needed to certify finite convergence in [Nie14], the moment relaxation has the flat truncation property at some order, see Theorem 3.5.7.

3.1.2 Structure of the chapter

This chapter is structured as follows.

- Section 3.1 describes the context and the results of the chapter. In Section 3.1.1 we present the related literature and compare it with our results, while Section 3.1.2 describes the structure of the chapter.
- Section 3.2 presents basic properties of Lasserre's hierarchies related to finite convergence, and introduced the basic definitions of the chapter.
- Section 3.3 is devoted to the formal definition of finite convergence and exactness for the sum of square and moment hierarchies, and to the presentation of basic examples in connection with flat truncation. In Section 3.3.1 we show how these notions are (and are not) related, presenting several new examples.
- Section 3.4 develops the theoretical tools needed for the remaining of the chapter. The properties of positive truncated pseudo-moment sequences are described in Section 3.4.1, while Section 3.4.2 describe the annihilators of those sequences (i.e., the kernel of the moment matrix). We then analyze when flat truncation holds and relate it with the regularity of S in Section 3.4.3.
- Section 3.5 applies the previous results to polynomial optimization. In Section 3.5.1 we prove the main result of the chapter, the equivalence between flat truncation and zero dimensionality of the support of $Q + (f - f^*)$. In Section 3.5.2 we apply the previous result to show that flat truncation and exactness are generic properties. Finally, we prove that exactness and flat truncation hold for finite semialgebraic sets (Section 3.5.3), and in particular with the addition of the polar constraints (Section 3.5.4).
- Section 3.6 is devoted to the presentation of some examples of the optimal power flow problem.
- Section 3.7 concludes the chapter, highlighting possible research directions.

3.2 Basic properties of Lasserre's hierarchies

We recall Lasserre's SoS and moment relaxations [Las01], introduced in section 1.6:

$$f^*_{\mathrm{SoS},d} = \sup\left\{ \lambda \in \mathbb{R} \mid f - \lambda \in \mathcal{Q}_{2d}(\mathbf{g}) \right\}$$

$$f^*_{\mathrm{Mom},d} = \inf\left\{ \langle \Lambda | f \rangle \in \mathbb{R} \mid \Lambda \in \mathcal{L}^{(1)}_{2d}(\mathbf{g}) \right\}$$

where $f, g_1, \dots g_r \in \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \dots, x_n]$.

Recall also that $f^*_{\mathrm{SoS},d} \le f^*_{\mathrm{Mom},d} \le f^* = \inf_{x \in \mathcal{S}(\mathbf{g})} f(x)$. When necessary we will replace the tuple of constrains $\mathbf{g}$ by the tuple of their products $\Pi\mathbf{g}$ (that is $\mathcal{Q}(\mathbf{g})$ by $\mathcal{O}(\mathbf{g})$): this does not change the initial polynomial optimization problem, since $\mathcal{S}(\mathbf{g}) = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_r(x) \geq 0\} = \mathcal{S}(\Pi\mathbf{g})$.

Hereafter we assume that the infimum f^* of the objective function f is always attained on S, that is: $S^{\min} := \{x \in S \mid f(x) = f^*\} \neq \emptyset$. In particular this condition is satisfied when $\mathcal{Q}(\mathbf{g})$ is Archimedean (Definition 1.1.20), since in this case S is compact. Moreover, if $g_1 = r^2 - \|\mathbf{x}\|_2^2$ the infimum $f^*_{\mathrm{Mom},d}$ is reached. Indeed, $f^*_{\mathrm{SoS},d} > -\infty$ since $f - \lambda \in \mathcal{Q}(\mathbf{g})$ when λ is small enough, as a consequence of Putinar's Positivstellensatz and $f^*_{\mathrm{Mom},d} \geq f^*_{\mathrm{SoS},d} > -\infty$. Finally, consider the continuous function

$$\phi_f : \mathbb{R}[\mathbf{x}]^*_{2d} \to \mathbb{R}$$
$$\Lambda \mapsto \langle \Lambda | f \rangle$$

Therefore $f^*_{\mathrm{Mom},d} = \inf \phi_f(\mathcal{L}^{(1)}_{2d}(\mathbf{g}))$, and since $g_1 = r^2 - \|\mathbf{x}\|_2^2$ implies that $\mathcal{L}^{(1)}_{2d}(\mathbf{g})$ is bounded [JH16, lem. 3] and thus compact, $\phi_f(\mathcal{L}^{(1)}_{2d}(\mathbf{g}))$ is compact and $f^*_{\mathrm{Mom},d}$ is attained. When the quadratic module is Archimedean but the ball constraint is not explicit, we can still claim that $f^*_{\mathrm{SoS},d}$ is attained for d big enough, replacing [JH16, lem. 3] with Lemma 2.5.8.

There are examples where the natural properties just described can fail, when the assumptions above are not satisfied.

Example 3.2.1. [[Mar09, ex. 5.2]] Let $f = 1 - 3x^2y^2 + x^4y^2 + x^2y^4$ be the Motzkin polynomial, that we minimize globally on $\mathbb{R}^n$. In this case $f^* = 0$, and the four minimizers are $(\pm 1, \pm 1)$. Since $f \in \mathrm{Pos}(\mathbb{R}^n) \setminus \Sigma^2$, then $f - \lambda \notin \mathcal{Q}_{2d}(1) = \Sigma^2_{2d} = \Sigma^2 \cap \mathbb{R}[\mathbf{x}]_{2d}$ for all $\lambda \in \mathbb{R}$ and $f^*_{\mathrm{SoS},d} = -\infty$ for all d. Furthermore, since $\mathcal{Q}_{2d}(1)$ is closed we can deduce that $f^*_{\mathrm{Mom},d} = -\infty$ from classical convexity arguments, as follows. Since Σ^2_{2d} is closed, $((\Sigma^2_{2d})^\vee)^\vee = \Sigma^2_{2d}$ from Conic Duality, Corollary 1.2.3. Therefore, since $f - \lambda \notin \Sigma^2_{2d}$ for all $\lambda \in \mathbb{R}$, then the Separation theorem Theorem 1.2.1 implies the existence of $\Lambda = \Lambda_\lambda \in \mathcal{L}_{2d}(1) = (\Sigma^2_{2d})^\vee$ such that $\langle \Lambda | f - \lambda \rangle < 0$. There are two cases:

- $\langle \Lambda | 1 \rangle > 0$. In this case, $\frac{\Lambda}{\langle \Lambda | 1 \rangle} \in \mathcal{L}^{(1)}_{2d}(1)$ and $\left\langle \frac{\Lambda}{\langle \Lambda | 1 \rangle} \middle| f \right\rangle < \lambda$.
- $\langle \Lambda | 1 \rangle = 0$. In this case, since $-1 \notin \Sigma^2_{2d}$, there exists $\eta \in \mathcal{L}_{2d}(1)$ such that $\langle \eta \big| 1 \rangle < 0$, and thus $\langle \eta \big| 1 \rangle = -\langle \eta \big| -1 \rangle > 0$. Scaling, we can assume $\langle \eta \big| 1 \rangle = 1$. Therefore, for all $r \in \mathbb{R}_{\geq 0}$, $r\Lambda + \eta \in \mathcal{L}^{(1)}_{2d}(1)$ and $\langle r\Lambda + \eta \big| f - \lambda \rangle = r \langle \Lambda | f - \lambda \rangle + \langle \eta \big| f - \lambda \rangle < 0$, for r big enough.

In both cases, we have constructed $\Lambda = \Lambda_\lambda \in \mathcal{L}^{(1)}_{2d}(1)$ such that $\langle \Lambda | f \rangle < \lambda$. Therefore, $f^*_{\mathrm{Mom},d} < \lambda$ for all $\lambda \in \mathbb{R}$, and then $f^*_{\mathrm{Mom},d} = -\infty$. In particular, $f^*_{\mathrm{SoS},d}$ and $f^*_{\mathrm{Mom},d}$ are not attained.

In Example 3.2.2, we show that $\mathcal{L}^{(1)}_{2d}(\mathbf{g})$ can be unbounded even in the case when $f^*_{\mathrm{SoS},d}$ and $f^*_{\mathrm{Mom},d}$ are finite and attained and the quadratic module $\mathcal{Q}(\mathbf{g})$ is Archimedean.

Example 3.2.2. [[Nie13b, ex. 1.1]] Let $f = x^3$ that we minimize on $[0,1] = \mathcal{S}(x, 1-x)$. Since $x^3 = x^2 x \in \mathcal{Q}_3(x, 1-x)$, then $f^*_{\mathrm{SoS},d} = f^*_{\mathrm{Mom},d} = f^* = 0$ for all $d \geq 2$. However, one can show that linear functionals $\Lambda_M \in \mathbb{R}[x]^*_{2d}$ with pseudo-moments:

$$\Lambda_0 = 1, \Lambda_1 = 0, \ldots, \Lambda_{2d-1} = 0, \Lambda_{2d} = M$$

belongs to $\mathcal{L}^{(1)}_{2d}(x, 1-x)$ for all $M \geq 0$. This shows that $\mathcal{L}^{(1)}_{2d}(x, 1-x)$ is not bounded and, in particular, not compact.

The problem in Example 3.2.2 is that the ball constraint in not explicit in the description of $\mathcal{S}(\mathbf{g})$, and thus we cannot apply [JH16, lem. 3]. However, if we only know that $\mathcal{Q}(\mathbf{g})$ is Archimedean, we can still apply Lemma 2.5.8: then, when d is big enough $\mathcal{L}_{2d}^{(1)}(\mathbf{g})^{[\deg f]}$ is bounded, compact and thus

$$f^*_{\mathrm{Mom},d} = \inf\left\{ \left\langle \Lambda^{[\deg f]} \middle| f \right\rangle \in \mathbb{R} \mid \Lambda \in \mathcal{L}_{2d}^{(1)}(\mathbf{g}) \right\} = \inf\left\{ \langle \Lambda | f \rangle \in \mathbb{R} \mid \Lambda \in \mathcal{L}_{2d}^{(1)}(\mathbf{g})^{[\deg f]} \right\}$$

is attained.

In the following we are interested, in particular, in linear functionals that realize the infimum $f^*_{\mathrm{Mom},d}$ (that we have seen to be attained, at least in the Archimedean case when, d is big enough).

Definition 3.2.3. Consider the problem of minimizing $f \in \mathbb{R}[\mathbf{x}]$ on $\mathcal{S}(\mathbf{g})$. We define the set of *functional minimizers* at relaxation order d as the Λ minimizing (1.7), i.e.:

$$\mathcal{L}_{2d}^{\min}(\mathbf{g}) := \left\{ \Lambda \in \mathcal{L}_{2d}^{(1)}(\mathbf{g}) \mid \langle \Lambda | f \rangle = f^*_{\mathrm{Mom},d} \right\}.$$

We give now a geometrical interpretation for the construction of $\mathcal{L}_{2d}^{\min}(\mathbf{g})$. We start from the spectrahedral cone:

$$\mathcal{L}_{2d}(\mathbf{g}) = \mathcal{Q}_{2d}(\mathbf{g})^\vee = \{ \Lambda \in \mathbb{R}[\mathbf{x}]^*_{2d} \mid H_\Lambda^{\lfloor \frac{d}{2} \rfloor} \succcurlyeq 0, H^{N_1}_{g_1 \star \Lambda} \succcurlyeq 0, \dots, H^{N_r}_{g_r \star \Lambda} \succcurlyeq 0 \},$$

where $N_i = \lfloor \frac{d - \deg g_i}{2} \rfloor$. Then we consider the section $\mathcal{L}_{2d}^{(1)}(\mathbf{g})$ of $\mathcal{L}_{2d}(\mathbf{g})$ given by $\langle \Lambda | 1 \rangle = 1$. This section is a convex, spectrahedral set, and finally the equation $\langle \Lambda | f \rangle = f^*_{\mathrm{Mom},d}$ defines $\mathcal{L}_{2d}^{\min}(\mathbf{g})$ as an exposed face of $\mathcal{L}_{2d}^{(1)}(\mathbf{g})$.

Now, from the discussion above we have seen that $\mathcal{L}_{2d}^{(1)}(\mathbf{g})$ may not be bounded. But we have also seen that the problem can be solved, at lest in the Archimedean case, letting d grow and projecting to $\mathcal{L}_{2d}^{(1)}(\mathbf{g})^{[k]}$, $k \leq 2d$. The same considerations work for $\mathcal{L}_{2d}^{\min}(\mathbf{g}) \subset \mathcal{L}_{2d}^{(1)}(\mathbf{g})$ (for instance, the unbounded family $\{\Lambda_M\}_{M \in \mathbb{R}_{\geq 0}}$ defined in Example 3.2.2 is a subset of $\mathcal{L}_{2d}^{\min}(\mathbf{g})$, that is therefore unbounded). We will then focus our attention on the projections $\mathcal{L}_{2d}^{\min}(\mathbf{g})^{[k]}$.

It is important to study the convex sets $\mathcal{L}_{2d}^{\min}(\mathbf{g})$ and their projections $\mathcal{L}_{2d}^{\min}(\mathbf{g})$ in order to understand the moment relaxations, since interior point solvers (`Mosek`, `SDPA`, `SEDUMI`, …), used to solve the semidefinite program associated to Lasserre's relaxations, output a truncated pseudo-moment sequence representing a linear functional lying in the relative interior of $\mathcal{L}_{2d}^{\min}(\mathbf{g})$.

3.3 Finite convergence and exactness

We now introduce two convergence properties that are the central topic of this chapter: *finite convergence* and *exactness* for the SoS and moment hierarchies.

Definition 3.3.1 (Finite Convergence)**.** We say that the SoS hierarchy $(\mathcal{Q}_{2d}(\mathbf{g}))_{d \in \mathbb{N}}$ (respectively, the moment hierarchy $(\mathcal{L}_{2d}(\mathbf{g}))_{d \in \mathbb{N}}$) has the *Finite Convergence* property for f if $\exists k \in \mathbb{N}$ such that for every $d \geq k$, $f^*_{\mathrm{SoS},d} = f^*$ (respectively, $f^*_{\mathrm{Mom},d} = f^*$).

Notice that if the SoS hierarchy has finite convergence then the moment hierarchy has finite convergence too, since $f^*_{\mathrm{SoS},d} \le f^*_{\mathrm{Mom},d} \le f^*$. Moreover, by definition if $f^*_{\mathrm{Mom},d} = f^*$ then $\mathcal{L}^{\min}_{2d}(\mathbf{g}) = \{\Lambda \in \mathcal{L}^{(1)}_{2d}(\mathbf{g}) \mid \langle \Lambda | f \rangle = f^*\}$.

In the definition of SoS relaxations, we have a sup and not a max. However, in practice one wants a *certificate of positivity* for $f - f^*$ on S, given by the representation $f - f^* = s_0 + s_1 g_1 + \dots s_r g_r \in \mathcal{Q}_{2d}(\mathbf{g})$ for some d.

Example 3.3.2. We give an example of such a certificate of positivity. Consider the square $[-1,1]^2 = \mathcal{S}(\mathbf{g})$ defined by the two inequalities $g_1 = (1+x)(1-y)$ and $g_2 = (1-x)(1+y)$. We want to find a certificate of positivity for $f = x^2 - x^3 y$ (notice that $f^* = 0$ is attained at $(1,1)$, $(-1,-1)$ and on the segment $\mathrm{conv}((0,-1),(0,1))$).

As explained in Section 1.3.7, we can search for a representation of $f \in \mathcal{Q}_{2d}(\mathbf{g})$ giving a positivity certificate using semidefinite programming. We use `MomentTools.jl` to search for this representation, choosing $d = 2$:

```
X = @polyvar x y
s, P, Q, v, M = sos_decompose(x^2-x^3*y,[0], [(1+x)*(1-y), (1-x)*(1+y)],X,2)
println(s)
-3.3564262480467733e-12x^4 - 7.708181115617663e-8x^3y + 8.084570390920476e-8x^2y^2 +
5.432637095736936e-8xy^3 + 1.1022738277688404e-11y^4 + 8.786447169839562e-8x^2 -
9.980950466115246e-8xy - 2.595994574861038e-8y^2 - 2.0188295923873056e-8

println(Q)
0.4999999614606291 4x^2 + 4.042666886247556e-8xy + 2.716147755799068e-8y^2 -
5.3929536263862865e-9x + 1.4184810360464893e-8y + 1.009489264624833e-8

0.49999996146062914x^2 + 4.042666886247556e-8xy + 2.716147755799068e-8y^2 +
5.3929536263862865e-9x - 1.4184810360464893e-8y + 1.009489264624833e-8

println(s + Q[1]*(1+x)*(1-y)+Q[2]*(1-x)*(1+y))
-3.3564262480467733e-12x^4 - 1.0000000000030695x^3y - 7.63381574635725e-12x^2y^2 +
3.4158413880020907e-12xy^3 + 1.1022738277688404e-11y^4 + 0.9999999999998228x^2 +
9.575745004352883e-12xy - 6.611353558804644e-12y^2 + 1.489368623605011e-12
```

this gives the (approximate) certificate of positivity for $x^2 - x^3 y$ on $[-1,1]^2$:

$$x^2 - x^3 y = 0 + \frac{1}{2}x^2(1+x)(1-y) + \frac{1}{2}x^2(1-x)(1+y).$$

When the representation $f - f^* = s_0 + s_1 g_1 + \dots s_r g_r \in \mathcal{Q}_{2d}(\mathbf{g})$ is possible, we call the SoS hierarchy *exact*.

Definition 3.3.3 (SoS Exactness)**.** We say that the SoS hierarchy $(\mathcal{Q}_{2d}(\mathbf{g}))_{d\in\mathbb{N}}$ is *exact* for f if it has the finite convergence property and for all d big enough, we have $f - f^* \in \mathcal{Q}_{2d}(\mathbf{g})$ (in other words sup = max in the definition of $f^*_{\mathrm{SoS},d}$).

Obviously, if $f - f^* \in \mathcal{Q}_{2d}(\mathbf{g})$ then $f - f^* \in \mathcal{Q}_{2k}(\mathbf{g})$ for all $k \ge d$: this means that if $f - f^* \in \mathcal{Q}_{2d}(\mathbf{g})$ then the SoS hierarchy is exact. This property is not shared in general for corresponding definition for the moment hierarchy, i.e. *moment exactness*, see Definition 3.3.4 and Example 3.3.5.

On one hand, the SoS hierarchy is based on an inner approximation of polynomials with truncated quadratic modules. On the other hand, the moment hierarchy is based on the approximation of measures with truncated positive linear functionals. Therefore, the property that we want is to be able to represent linear functionals in $\mathcal{L}_{2d}^{\min}(\mathbf{g})$ as measures supported on S. However, we are not interested in all the moments, but rather in the truncation of the moment sequence in some degree. We make this truncation for two reasons:

- first, $\langle \Lambda | f \rangle = \left\langle \Lambda^{[\deg f]} \middle| f \right\rangle$, so we do not care about moments of degree $> \deg f$;
- second, the higher order moments may not share the good properties of the low degree ones, see for instance Example 3.2.2.

Therefore, we ask the property that every truncated functional minimizer is coming from a measure:

Definition 3.3.4 (Moment Exactness)**.** We say that the moment hierarchy $(\mathcal{L}_{2d}(\mathbf{g}))_{d\in\mathbb{N}}$ is *exact* for f on the basic closed semialgebraic set S if:

- it has the finite convergence property, and
- for every $k \in \mathbb{N}$, there exists $d = d(k) \in \mathbb{N}$ such that every truncated functional minimizer is coming from a probability measure supported on S, i.e. $\mathcal{L}_{2d}^{\min}(\mathbf{g})^{[k]} \subset \mathcal{M}^{(1)}(S)^{[k]}$.

If not specified, S will be the semialgebraic set $S = \mathcal{S}(\mathbf{g})$ defined by $\mathbf{g}$.

Moments exactness may be considered as a particular instance of the so-called *Moment Problem* (i.e. asking if $\Lambda \in \mathbb{R}[\mathbf{x}]^*$ is coming from a measure) or of the *Strong Moment Problem* (i.e. asking that the measure has a specified support). More precisely, moment exactness can be considered as a *Truncated Strong Moment Property* (since we are considering functionals restricted to polynomials up to a bounded degree). See Section 1.4 for a more detailed discussion.

Notice that in the definition of exactness we require the property $\mathcal{L}_{2d(k)}^{\min}(\mathbf{g})^{[k]} \subset \mathcal{M}^{(1)}(S)^{[k]}$ to hold for every k, and in general the fact that the property is verified for particular k does not imply that it is verified for every k, as Example 3.3.5 shows.

Example 3.3.5. Consider the problem of minimizing $f = 1$ on $S = \mathbb{R}^2$. First, recall that $\mathcal{Q}(1) = \Sigma^2 = \Sigma\mathbb{R}[\mathbf{x}]^2$ is stable in a strong sense: $\mathcal{Q}(1) \cap \mathbb{R}[\mathbf{x}]_{2d} = \mathcal{Q}_{2d}(1) = \Sigma_{n,2d}^2$ for all $d \in \mathbb{N}$.

Now, notice that for all $d \geq k$ we have $\mathcal{L}_k(1) \supset \mathcal{L}_d(1)^{[k]} \supset \mathcal{L}_{d+1}(1)^{[k]} \supset \mathcal{L}(1)^{[k]}$, by definition. We prove the converse inclusion for $k = 2$. Recall the solution of Hilbert's 17th problem: $\Sigma_{n,2d}^2 = \operatorname{Pos}(\mathbb{R}^n)_{2d}$ if and only if:

- $n = 1$ and d arbitrary;
- $n = 2$ and $d \leq 2$;
- n arbitrary and $d = 1$.

Then we observe that $\mathcal{Q}_2(1) = \Sigma_{2,2}^2 = \operatorname{Pos}(\mathbb{R}^2)_2$ and thus from Theorem 2.5.2 $\mathcal{L}_2(1) = (\Sigma_{2,2}^2)^\vee = \operatorname{Pos}(\mathbb{R}^2)_{\leq 2}^\vee = \mathcal{M}(\mathbb{R}^2)^{[2]}$. Finally, $\mathcal{L}_2^{\min}(1) = \mathcal{L}_2(1)$ since $f = 1$ and thus the inclusion $\mathcal{L}_2^{\min}(1)^{[k]} \subset \mathcal{M}^{(1)}(S)^{[k]}$ is satisfied for $k = 2$

On the other hand, $\Sigma_{2,6}^2 \subsetneq \operatorname{Pos}(\mathbb{R}^2)_6$ (for instance, the Motzkin polynomial M is such that $M \in \operatorname{Pos}(\mathbb{R}^2)_6 \setminus \Sigma_{2,6}^2$), and therefore

$$\mathcal{M}(\mathbb{R}^n)^{[6]} \subsetneq (\Sigma_{2,6}^2)^\vee \subset \mathcal{L}_d(1)^{[6]}$$

for all $d \geq 6$. Therefore, the moment hierarchy is not exact.

We show now an example where we have finite convergence and exactness of the SoS and the moment hierarchies.

Example 3.3.6. Consider the problem of minimizing $f = x^2$ on the semialgebraic set $S = \mathcal{S}(\mathbf{g}) = \mathcal{S}(1-x^2-y^2, x+y-1) \subset \mathbb{R}^2$ defined by $g_1 = 1-x^2-y^2$ and $g_2 = x+y-1$. Clearly, the minimum is $f^* = 0$ and the only minimizer is $(0,1)$. Notice that $f - f^* = x^2 \in \mathcal{Q}_2(1-x^2-y^2, x+y-1)$ and therefore $f^*_{\mathrm{SoS},1} = f^*_{\mathrm{Mom},1} = f^* = 0$, we have finite convergence and the SoS hierarchy is exact.

We now investigate moment exactness. If a truncated moment sequence Λ is coming from a probability measure $\mu \in \mathcal{M}^{(1)}(S)$ such that $\int f \, \mathrm{d}\mu = f^*$, then the support of μ should be contained in the set of minimizers $S^{\min} = \{(0,1)\}$ of f. Thus, $\mu = \mathbf{e}_{(0,1)}$ is the evaluation at $(0,1)$ (or in other words, the Dirac measure concentrated at $(0,1)$). Its moments are easily computed: $\mu_{00} = 1$, $\mu_{10} = 0$, $\mu_{01} = 1$, $\mu_{20} = 0, \ldots$.

Analyzing the constraints on the degree one and two moments of an optimal moment sequence $\Lambda \in \mathcal{L}_2^{\min}(\mathbf{g})$, where

$$\begin{aligned}\mathcal{L}_2^{\min}(\mathbf{g}) &= \{\Lambda \in \mathbb{R}[\mathbf{x}]_2^* \mid H_\Lambda^1 \succcurlyeq 0,\ H^0_{g_1\star\Lambda} \succcurlyeq 0,\ H^0_{g_2\star\Lambda} \succcurlyeq 0, \langle\Lambda|1\rangle = 1, \langle\Lambda|f\rangle = f^* = 0\} \\ &= \{\Lambda \in \mathbb{R}[\mathbf{x}]_2^* \mid \begin{pmatrix}\Lambda_{00} & \Lambda_{10} & \Lambda_{01}\\ \Lambda_{10} & \Lambda_{20} & \Lambda_{11}\\ \Lambda_{01} & \Lambda_{11} & \Lambda_{02}\end{pmatrix} \succcurlyeq 0,\ \Lambda_{00} - \Lambda_{20} - \Lambda_{02} \geq 0,\ \Lambda_{10} + \Lambda_{01} - \Lambda_{00} \geq 0, \Lambda_{00} = 1, \Lambda_{20} = 0\},\end{aligned}$$

we deduce that $\Lambda_{00} = 1$, $\Lambda_{10} = 0$, $\Lambda_{01} = 1$, $\Lambda_{20} = 0$, $\Lambda_{11} = 0$ and $\Lambda_{02} = 1$: this shows that the only element of $\mathcal{L}_2^{\min}(\mathbf{g})$ is $\Lambda = \mathbf{e}^{[2]}_{(0,1)}$. In particular notice that $\left\langle\Lambda\middle|x^2\right\rangle = \left\langle\Lambda\middle|(y-1)^2\right\rangle = 0$.

For any order $d \geq 1$ and any element $\Lambda \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$, its truncation $\Lambda^{[2]}$ is in $\mathcal{L}_2^{\min}(\mathbf{g})$ since $\left\langle\Lambda^{[2]}\middle|x^2\right\rangle = \left\langle\Lambda\middle|x^2\right\rangle = 0$ and $\left\langle\Lambda^{[2]}\middle|(y-1)^2\right\rangle = \left\langle\Lambda\middle|(y-1)^2\right\rangle = 0$ imply that $\forall p \in \mathbb{R}[\mathbf{x}]_d, \langle\Lambda|xp\rangle = \left\langle\Lambda\middle|(y-1)p\right\rangle = 0$, see Lemma 1.3.10. We deduce from Proposition 3.4.15 that the moments of $\Lambda^{[d]} = \mathbf{e}^{[d]}_{(0,1)}$ are coming from the Dirac measure $\mathbf{e}_{(0,1)}$. Therefore, the moment hierarchy is exact.

Another equivalent way to certify moment exactness is to check flat truncation (see Definition 3.4.17 and Section 1.4.1). For $\Lambda \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ with $d \geq 2$, we have computed the moments of degree ≤ 2. Since the moment matrices in degree ≤ 2:

$$H_\Lambda^0 = \begin{pmatrix}1\end{pmatrix}, \quad H_\Lambda^1 = \begin{pmatrix}1 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 1\end{pmatrix}$$

have the same rank, the flat extension property is satisfied. This certifies that $\Lambda^{[2]} = \mathbf{e}^{[2]}_{(0,1)}$ is coming from a measure supported at the minimizer of f on S and the moment hierarchy is exact, see Theorem 3.5.1.

In practice, to check the finite convergence, one tests the flat extension or the flat truncation property of moment matrices (see [CF98], [LM09], [Nie13b]). But flat truncation certifies moment exactness, and not only finite convergence. We will investigate flat truncation for polynomial optimization problems in Section 3.5.

Notice that in the previous example the rank condition is satisfied by the full sequence of moments of $\Lambda \in \mathcal{L}_2^{\min}(\mathbf{g})$ (i.e. we have *flat extension*). In general this is not true, as the high degree moments may be increasing the rank of the moment matrix, see for in instance Example 3.2.2. Therefore, it is necessary to discard the high degree moments, i.e. to consider $\mathcal{L}_{2d}^{\min}(\mathbf{g})^{[t]}$, for some $t \leq 2d$, instead of simply $\mathcal{L}_{2d}^{\min}(\mathbf{g})$. This implies that we look for rank conditions on the moment matrix of the truncated moment sequence (i.e. we have *flat truncation*).

We recall results of strong duality, that is cases when we know that $f^*_{\mathrm{SoS},d} = f^*_{\mathrm{Mom},d}$, that we will be using in the chapter. See also Proposition 3.4.8.

Theorem 3.3.7 (Strong duality). *Let $Q = \mathcal{Q}(\mathbf{g})$ be a quadratic module and f the objective function. Then:*

(i) if $\operatorname{supp} Q = 0$ then $\forall d$: $f^*_{\mathrm{SoS},d}$ is attained (i.e. $f - f^*_{\mathrm{SoS},d} \in \mathcal{Q}_d(\mathbf{g})$) and $f^*_{\mathrm{SoS},d} = f^*_{\mathrm{Mom},d}$[Mar08, prop. 10.5.1];

(ii) if $g_1 = r^2 - \|\mathbf{x}\|_2^2$ then $f^*_{\mathrm{SoS},d} = f^*_{\mathrm{Mom},d}$ for all d [JH16].

Remark. [JH16] applies when the ball constraint $r^2 - \|\mathbf{x}\|_2^2$ appears explicitly in the description of S. But if we consider a problem with moment finite convergence and such that $\mathcal{Q}(\mathbf{g})$ is Archimedean, then we can use [JH16] to prove that we have also SoS finite convergence. Indeed, if $\mathcal{Q}(\mathbf{g})$ is Archimedean there exists r, t such that $r^2 - \|\mathbf{x}\|_2^2 \in \mathcal{Q}_{2t}(\mathbf{g})$. This means that $\mathcal{Q}_{2d}(\mathbf{g}, r^2 - \|\mathbf{x}\|_2^2) \subset \mathcal{Q}_{2d+2t}(\mathbf{g})$. If we define:

- $f^*_{\mathrm{SoS},d} = \sup\left\{\lambda \in \mathbb{R} \mid f - \lambda \in \mathcal{Q}_{2d}(\mathbf{g})\right\}$
- $f^{*'}_{\mathrm{SoS},d} = \sup\left\{\lambda \in \mathbb{R} \mid f - \lambda \in \mathcal{Q}_{2d}(\mathbf{g}, r^2 - \|\mathbf{x}\|_2^2)\right\}$

and $f^*_{\mathrm{Mom},d}$, $f^{*'}_{\mathrm{Mom},d}$ the corresponding moment relaxations, then:

$$f^*_{\mathrm{Mom},d} \leq f^{*'}_{\mathrm{Mom},d} = f^{*'}_{\mathrm{SoS},d} \leq f^*_{\mathrm{SoS},d+t} \leq f^*.$$

Then finite convergence of the moment hierarchy implies finite convergence of the SoS one.

3.3.1 Examples and counterexamples

In this section, we give examples showing how the notions of finite convergence and exactness of the SoS and moment hierarchies are (and are not) related.

No finite convergence. The first example shows that SoS and moment hierarchies for polynomial optimization on algebraic curves do not have necessarily the finite convergence property.

Example 3.3.8 ([Sch00]). Let $\mathcal{C} \subset \mathbb{R}^n$ be a smooth connected curve of genus ≥ 1, with only real points at infinity. Let $\mathbf{h} = \{h_1, \ldots, h_s\} \subset \mathbb{R}[\mathbf{x}]$ be a graded basis of $I = \mathcal{I}(\mathcal{C}) = (\mathbf{h})$. Then there exists $f \in \mathbb{R}[\mathbf{x}]$ such that the SoS hierarchy $\mathcal{Q}_{2d}(\pm\mathbf{h})$ and the moment hierarchy $\mathcal{L}_{2d}(\pm\mathbf{h})$ have no finite convergence and are not exact.

Indeed, by [Sch00, Theorem 3.2], there exists $f \in \mathbb{R}[\mathbf{x}]$ such that $f \geq 0$ on $\mathcal{C} = \mathcal{S}(\pm\mathbf{h})$, which is not a sum of squares in $\mathbb{R}[\mathcal{C}] \cong \mathbb{R}[\mathbf{x}]/I$. Consequently, $f \notin \Sigma\mathbb{R}[\mathbf{x}]^2 + I = \mathcal{Q}(\pm\mathbf{h})$. As $f \geq 0$ on $\mathcal{C}$, its infimum f^* is non-negative, and we also have $f - f^* \notin \mathcal{Q}(\pm\mathbf{h})$.

Using Proposition 3.4.8 we deduce that $\mathcal{Q}_{2d}(\pm\mathbf{h})$ is closed, that there is no duality gap and that the supremum $f^*_{\mathrm{SoS},d}$ is reached. Thus, if the SoS hierarchy has finite convergence then $f - f^* \in \mathcal{Q}_{2d}(\pm\mathbf{h})$ for some $d \in \mathbb{N}$. This is a contradiction, showing that the SoS and the moment hierarchies have no finite convergence and cannot be SoS exact for f.

In dimension 2, there are also cases where the SoS and moment hierarchies cannot have finite convergence or be exact.

Example 3.3.9 ([Mar08]). Let $g_1 = x_1^3 - x_2^2, g_2 = 1 - x_1$. Then $S = \mathcal{S}(\mathbf{g})$ is a compact semialgebraic set of dimension 2 and $\mathcal{O}(\mathbf{g}) = \mathcal{Q}(g_1, g_2, g_1 g_2)$ is Archimedean. We have $f = x_1 \geq 0$ on S but $x_1 \notin \mathcal{O}(\mathbf{g})$ (see [Mar08, Example 9.4.6(3)]). The infimum of f on S is $f^* = 0$. Assume that we have moment finite convergence. By Theorem 3.3.7 and remark below, $\mathcal{Q}_{2d}(g_1, g_2, g_1 g_2)$ is closed, the supremum $f^*_{\mathrm{SoS},d}$ is reached and strong duality holds: $f^*_{\mathrm{SoS},d} = f^*_{\mathrm{Mom},d} = f^* = 0$ for all $d \in \mathbb{N}$ big enough. Then $f - f^* = f \in \mathcal{O}(\mathbf{g})$: but this is a contradiction. Therefore, the hierarchies $\mathcal{Q}_{2d}(\Pi\mathbf{g})$ and $\mathcal{L}_{2d}(\Pi\mathbf{g})$ cannot have finite convergence and thus cannot be exact for $f = x_1$.

The next example shows that non-finite convergence and non-exactnesss is always possible in dimension ≥ 3.

Example 3.3.10. Let $n \geq 3$. Let Q be an Archimedean quadratic module generated by $g_1, \ldots, g_s \in \mathbb{R}[\mathbf{x}]$ such that $\mathcal{S}(\mathbf{g}) \subset \mathbb{R}^n$ is of dimension $m \geq 3$ and let $\mathbf{h}$ be a graded basis of $\sqrt[\mathbb{R}]{\operatorname{supp} Q}$ (in particular $\mathbf{h} = 0$ if $\operatorname{supp} Q = 0$ or if $m = n$, that is if $\mathcal{S}(\mathbf{g})$ is of maximal dimension), then there exists $f \in \mathbb{R}[\mathbf{x}]$ such that the SoS hierarchy $(\mathcal{Q}_{2d}(\mathbf{g}, \pm\mathbf{h}))_{d\in\mathbb{N}}$ and moment hierarchy $(\mathcal{L}_{2d}(\mathbf{g}, \pm\mathbf{h}))_{d\in\mathbb{N}}$ do not have the finite convergence property (and thus are not exact).

Indeed by Proposition 3.4.8 $f^*_{\mathrm{SoS},d} = f^*_{\mathrm{Mom},d}$ for d big enough and the supremum $f^*_{\mathrm{SoS},d}$ is reached. By [Sch00, Prop. 6.1] for $m \geq 3$, $\operatorname{Pos}(\mathcal{S}(\mathbf{g})) = \operatorname{Pos}(\mathcal{S}(Q + (\mathbf{h}))) \supsetneq Q + (\mathbf{h})$. So let $f \in \operatorname{Pos}(\mathcal{S}(Q)) \setminus Q + (\mathbf{h})$ and let f^* be its minimum on $\mathcal{S}(Q)$. Suppose that $f - f^* \in Q + (\mathbf{h})$, then $f \in Q + (\mathbf{h}) + f^* = Q + (\mathbf{h})$, a contradiction. Then the SoS and the moment hierarchies do not have the finite convergence property (and they are not exact).

SoS exactness, no moment exactness.

Example 3.3.11. We want to find the global minimum of $f = x_1^2 \in \mathbb{R}[x_1, \ldots, x_n] = \mathbb{R}[\mathbf{x}]$ for $n \geq 3$. Let $d \geq 2$, $\mathbf{x}' = (x_2, \ldots, x_n)$ and $\overline{\Lambda} \in \mathcal{L}_{2d}(1)_{\mathbf{x}'} = (\Sigma(\mathbb{R}[\mathbf{x}']_d)^2)^\vee$ such that $\overline{\Lambda} \notin \mathcal{M}(\mathbb{R}^{n-1})^{[d]}$. Such a linear functional exists because when $n > 2$ there are non-negative polynomials in $\mathbb{R}[\mathbf{x}']$ which are not sum of squares, such as the Motzkin polynomial (see [Rez96]). As $\Sigma(\mathbb{R}[\mathbf{x}']_d)^2 = \Sigma\mathbb{R}[\mathbf{x}']^2 \cap \mathbb{R}[\mathbf{x}']_{2d}$ is closed, such a polynomial can be separated from $\Sigma(\mathbb{R}[\mathbf{x}']_d)^2$ by a linear functional $\overline{\Lambda} \in \mathcal{L}_{2d}(1)_{\mathbf{x}'} = (\Sigma(\mathbb{R}[\mathbf{x}']_d)^2)^\vee$, which cannot be the truncation of a measure. Define $\Lambda : h \mapsto \langle \Lambda | h \rangle := \left\langle \overline{\Lambda} \middle| h(0, x_2, \ldots, x_n) \right\rangle$. We have $\Lambda \in \mathcal{L}_{2d}(1) = (\Sigma\mathbb{R}[\mathbf{x}]^2 \cap \mathbb{R}[\mathbf{x}]_{2d})^\vee$ since $\overline{\Lambda} \in \mathcal{L}_{2d}(1)_{\mathbf{x}'}$. Obviously $\langle \Lambda | f \rangle = 0 = f^*$ (the minimum of x_1^2), $f - f^* = x_1^2 \in \Sigma^2$ and the SoS relaxation is exact. Since Λ is coming from a measure if and only if $\overline{\Lambda}$ is coming from a measure, the moment hierarchy cannot be exact.

The previous example generalizes easily to quadratic modules $Q = \mathcal{Q}(\mathbf{g})$ with $\operatorname{supp}(Q) \neq \{0\}$, which do not have the (truncated) moment property, i.e. there exists $\Lambda \in \mathcal{L}_{2d}(\mathbf{g})$ such that $\Lambda \notin \mathcal{M}(\mathcal{S}(Q))^{[2d]}$. Taking $f = h^2$ with $h \in \operatorname{supp}(Q)$, $h \neq 0$, we have $\langle \Lambda | f \rangle = 0 = f^*$ and the moment hierarchy cannot be exact since $\Lambda \notin \mathcal{M}(\mathcal{S}(Q))^{[2d]}$, while the SoS hierarchy is exact $(f - f^* = h^2 \in Q)$.

Example 3.3.11 is an example where the number of minimizers of f on S is infinite. We show that non exactness can happen also when the minimizers are finite (and even when S is finite!).

Example 3.3.12 ([Sch05a, ex. 3.2], [Sch05b, rem. 3.15], Example 3.4.4)**.** We want to minimize the constant function $f = 1$ on the origin $S = \mathcal{S}(Q) = \{\mathbf{0}\}$, where $Q = \mathcal{Q}(1 - x^2 - y^2, -xy, x - y, y - x^2) \subset \mathbb{R}[x,y]$. In this case $\operatorname{supp} Q = \sqrt[\mathbb{R}]{\operatorname{supp} Q} = (0)$. Notice that the SoS hierarchy is exact and the moment hierarchy has finite convergence, since f is a square. Now suppose that the moment hierarchy is exact, i.e. $\mathcal{L}_{2d}^{\min}(\mathbf{g})^{[2k]} = \mathcal{L}_{2d}^{(1)}(\mathbf{g})^{[2k]} \subset \mathcal{M}^{(1)}(S)^{[2k]} = \{\mathbf{e}_{\mathbf{0}}^{[2k]}\}$ for some d,k. Then for $\Lambda^* \in \mathcal{L}_{2d}(\mathbf{g})$ generic (see Definition 3.4.9), we have $(\Lambda^*)^{[2k]} = \lambda \mathbf{e}_{\mathbf{0}}^{[2k]}$ and $(\operatorname{Ann}_k(\Lambda^*)) = (\operatorname{Ann}_k(\mathbf{e_0})) = (x,y)$. But from Theorem 3.4.11 we know that for d,k big enough $(\operatorname{Ann}_k(\Lambda^*)) = \sqrt[\mathbb{R}]{\operatorname{supp} Q} = (0)$, a contradiction. Then the moment hierarchy is not exact. Moreover, the flat truncation property is not satisfied in this case: see Theorem 3.5.4.

We investigate concretely this example for $d = 1$. We show in Figure 3.1[1] the plot of $\mathcal{L}_2(\mathbf{g})^{[1]}$, that is, the pseudo-moments of degree one of the linear functionals $\mathcal{L}_2(\mathbf{g}) = \mathcal{Q}_2(\mathbf{g})^\vee$. Notice that this is an outer approximation of $\mathbf{e}_{(0,0)} \in \mathcal{L}_2(\mathbf{g})^{[1]}$ or, identifying moments of degree one with points of $\mathbb{R}^n$, a convex outer approximation of $S = \{(0,0)\}$.

One can also verify explicitly that $\mathcal{L}_2(\mathbf{g})$ has nonempty interior. Indeed, $\mathcal{L}_2(\mathbf{g})$ is explicitly defined by the following matrix inequalities:

$$\begin{aligned}
H_\Lambda^2 &= \begin{pmatrix} \Lambda_{00} & \Lambda_{10} & \Lambda_{01} \\ \Lambda_{10} & \Lambda_{20} & \Lambda_{11} \\ \Lambda_{01} & \Lambda_{11} & \Lambda_{02} \end{pmatrix} && \succcurlyeq 0, \\
H^0_{(1-x^2-y^2)\star\Lambda} &= \begin{pmatrix} 1 - \Lambda_{20} - \Lambda_{02} \end{pmatrix} && \succcurlyeq 0, \\
H^0_{-xy\star\Lambda} &= \begin{pmatrix} -\Lambda_{11} \end{pmatrix} && \succcurlyeq 0, \\
H^0_{(x-y)\star\Lambda} &= \begin{pmatrix} \Lambda_{10} - \Lambda_{01} \end{pmatrix} && \succcurlyeq 0, \\
H^0_{(y-x^2)\star\Lambda} &= \begin{pmatrix} \Lambda_{01} - \Lambda_{20} \end{pmatrix} && \succcurlyeq 0
\end{aligned}$$

Since $\Lambda = \Lambda(\varepsilon)$ defined by $\Lambda_{10} = 2\varepsilon$, $\Lambda_{01} = \varepsilon$, $\Lambda_{20} = \frac{\varepsilon}{2}$, $\Lambda_{11} = -\varepsilon^2$ and $\Lambda_{02} = \frac{1}{2}$ strictly satisfies those inequalities for $\varepsilon > 0$ small enough, $\Lambda(\varepsilon)$ lies in the interior of $\mathcal{L}_2(\mathbf{g})$ for $\varepsilon > 0$ small enough.

Notice that $\mathcal{L}_2(\mathbf{g})^{[1]} \supset \mathcal{L}_3(\mathbf{g})^{[1]} \supset \mathcal{L}_4(\mathbf{g})^{[1]} \supset \cdots \supset \{\mathbf{e}_{(0,0)}^{[1]}\}$, and we have convergence in this case since $\mathcal{Q}(\mathbf{g})$ is Archimedean, see Section 2.5. This nested outer approximations, shown in Figure 3.1[2], never coincide with $\{\mathbf{e}_{(0,0)}^{[1]}\}$, as we have proven before.

SoS finite convergence, moment exactness.

Example 3.3.13. Let $f = (x^4y^2 + x^2y^4 + z^6 - 2x^2y^2z^2) + x^8 + y^8 + z^8 \in \mathbb{R}[x,y,z]$. We want to optimize f over the gradient variety $\mathcal{V}_{\mathbb{R}}\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ which is zero dimensional (see [NDS06]). By Theorem 3.5.11 the flat truncation is satisfied and the moment hierarchy is exact, and by Theorem 3.3.7 and remark below the SoS has the finite convergence property (notice that $\mathcal{Q}(\pm\frac{\partial f}{\partial x}, \pm\frac{\partial f}{\partial y}, \pm\frac{\partial f}{\partial z}) = \mathcal{O}(\pm\frac{\partial f}{\partial x}, \pm\frac{\partial f}{\partial y}, \pm\frac{\partial f}{\partial z}) = \left(\frac{\partial f}{\partial x}, \pm\frac{\partial f}{\partial y}, \pm\frac{\partial f}{\partial z}\right) + \Sigma^2$ is Archimedean since

[1] The variables x,y in the plots, done using SDPA, have been scaled by 100 to reduce floating points errors

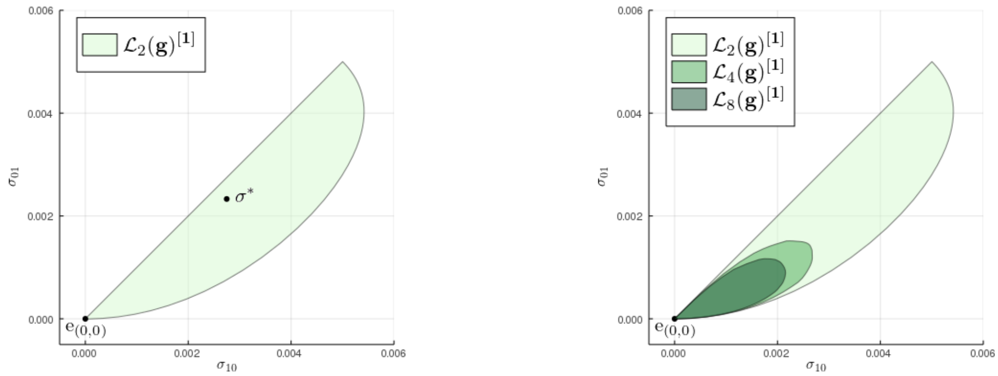

Figure 3.1: A generic point $\Lambda^* \in \mathcal{L}_2^{(1)}(\mathbf{g})^{[1]}$ and outer approximations of $\mathcal{L}^{(1)}(\mathbf{g})^{[1]} = \{\mathbf{e}_{0,0}^{[1]}\}$.

Table 3.1: Summary of convergence results.

Expl.	SoS f. c.	SoS ex.	Mom. f. c.	Mom ex.	Flat t.	m
3.3.8	NO	NO	NO	NO	NO	1
3.3.9	NO	NO	NO	NO	NO	2
3.3.10	NO	NO	NO	NO	NO	≥ 3
3.3.11	YES	YES	YES	NO	NO	≥ 3
3.3.12	YES	YES	YES	NO	NO	0
3.3.13	YES	NO	YES	YES	YES	0
3.3.14	YES	NO	YES	YES	YES	0

$\mathcal{V}_{\mathbb{R}}\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ is compact, from Schmüdgen's Positivstellensatz). But the SoS hierarchy is not exact, as shown in [NDS06].

Example 3.3.14. Let $f = x_1$. We want to find its value at the origin, defined by $\|\mathbf{x}\|_2^2 = 0$. As proved in [Nie13c] there is finite convergence but not exactness for the SoS hierarchy. On the other hand by Theorem 3.5.11 the flat truncation property is satisfied and the moment hierarchy is exact. This example is essentially Example 1.3.4.

We summarize the previous examples in Table 3.1 in terms of the properties of finite convergence (SoS f.c. and moment f.c.) exactness (SoS ex. and moment ex.), flat truncation, and the dimension m of the semialgebraic set S.

3.4 Geometry of pseudo-moment representations

Motivated from the study of the Lasserre's moment hirarchy, and in particular from the properties of finite convergence, exactness and flat truncation, we analyze in this section the properties of finite dimensional *truncated* cones of positive linear functionals $\mathcal{L}_d(\mathbf{g})$ (or, equivalently, properties of the representing pseudo-moment sequences). We provide a new and explicitly description of the dual of the hierarchy of $\mathcal{L}_d(\mathbf{g})$, in terms of a quadratic module (Theorem 3.4.3), and consequently prove properties of the cones $\mathcal{L}_d(\mathbf{g})$ (Lemma 3.4.5) and of their generic elements (Theorem 3.4.11). Genericity is an important property, as linear functionals in the relative interior of our convex sets are generic. Finally, we apply these results to the zero dimensional case (Theorem 3.4.20) and we investigate the connections with the flat truncation property (Section 3.4.3).

3.4.1 Truncated pseudo-moment representations

For a finitely generated quadratic module $Q = \mathcal{Q}(\mathbf{g}) \subset \mathbb{R}[\mathbf{x}]$, we have $\mathcal{L}_k(\mathbf{g}) = \mathcal{Q}_k(\mathbf{g})^\vee = \overline{\mathcal{Q}_k(\mathbf{g})}^\vee$ and $\mathcal{L}_k(\mathbf{g})^\vee = \overline{\mathcal{Q}_k(\mathbf{g})}$, where $^\vee$ denotes the dual cone and the closure is taken w.r.t. the euclidean topology on $\mathbb{R}[\mathbf{x}]_k$, see Section 1.2.4. See also Section 1.6.4 for more motivations to study closures of truncated quadratic modules in polynomial optimization. Thus, the following definition is natural for the study of moment relaxations.

Definition 3.4.1. Let $Q = \mathcal{Q}(\mathbf{g})$ be a finitely generated quadratic module. We define $\widetilde{Q} = \bigcup_d \overline{\mathcal{Q}_d(\mathbf{g})}$.

Notice that $\widetilde{Q}$ depends a priori on the generators $\mathbf{g}$ of Q: we will prove that $\widetilde{Q}$ is a finitely generated quadratic module and that it does not depend on the particular choice of generators. Moreover notice that $Q \subset \widetilde{Q} = \bigcup_d \overline{\mathcal{Q}_d(\mathbf{g})} \subset \overline{\bigcup_d \mathcal{Q}_k(\mathbf{g})} = \overline{Q}$, but these inclusions can be strict.

Lemma 3.4.2. *Let $Q = \mathcal{Q}(\mathbf{g})$ and $J = \sqrt[\mathbb{R}]{\operatorname{supp} Q}$. Then for every $d \in \mathbb{N}$ there exists $k \geq d$ such that $J_d \subset \overline{\mathcal{Q}_k(\mathbf{g})}$.*

Proof. We denote $\mathcal{Q}_d(\mathbf{g}) =: Q_{[d]}$. Let m be big enough such that $\forall f \in J = \sqrt[\mathbb{R}]{\operatorname{supp} Q} = \sqrt{\operatorname{supp} Q}$ (see Lemma 1.1.24) we have: $f^{2^m} \in \operatorname{supp} Q$ (if $\sqrt{J} = (h_1, \ldots, h_t)$ and $h_i^{a_i} \in I$, we can take m such that $2^m \geq a_1 + \cdots + a_t$). Let $f \in J_d$ with $\deg f \leq d$. Then $f^{2^m} \in \operatorname{supp} \mathcal{Q}_{[k']} \subset Q_{[k']}$ for $k' \in \mathbb{N}$ big enough. Using the identity [Sch05b, remark 2.2]:

$$m - a = (1 - \frac{a}{2})^2 + (1 - \frac{a^2}{8})^2 + (1 - \frac{a^4}{128})^2 + \cdots + (1 - \frac{a^{2^{m-1}}}{2^{2^m - 1}})^2 - \frac{a^{2^m}}{2^{2^{m+1}-2}},$$

substituting a by $-\frac{mf}{\varepsilon}$ and multiplying by $\frac{\varepsilon}{m}$, we have that $\forall \varepsilon > 0$, $f + \varepsilon \in Q_{[k]}$ for $k = \max\{k', 2^m d\}$ (the degree of the representation of $f + \varepsilon$ does not depend on ε). This implies that $f \in \overline{\mathcal{Q}_{[k]}}$. □

We can now describe $\widetilde{Q}$.

Theorem 3.4.3. *Let $Q = \mathcal{Q}(\mathbf{g})$ be a finitely generated quadratic module and let $J = \sqrt[\mathbb{R}]{\operatorname{supp} Q}$. Then $\widetilde{Q} = Q + J$ and $\operatorname{supp} \widetilde{Q} = J$. In particular, $\widetilde{Q}$ is a finitely generated quadratic module and does not depend on the particular choice of generators of Q.*

Proof. We denote $\mathcal{Q}_d(\mathbf{g}) =: Q_{[d]}$. By [Mar08, lemma 4.1.4] $Q_{[d]} + J_d$ is closed in $\mathbb{R}[\mathbf{x}]_d$, thus $\overline{Q_{[d]}} \subset Q_{[d]} + J_d$. Taking unions we prove that $\widetilde{Q} \subset Q + J$.

Conversely by Lemma 3.4.2 for $d \in \mathbb{N}$ and $k \geq d \in \mathbb{N}$ big enough, $J_d \subset \overline{Q_{[k]}}$. Then, we have $Q_{[d]} + J_d \subset Q_{[k]} + \overline{Q_{[k]}} \subset \overline{Q_{[k]}} + \overline{Q_{[k]}} \subset \overline{Q_{[k]}}$. Taking unions on both sides gives $Q + J \subset \widetilde{Q}$.

Finally $\operatorname{supp} \widetilde{Q} = \operatorname{supp}(Q + J) = J$ by [Sch05b, lemma 3.16]. □

Remark. We proved that $\widetilde{Q} = Q + \sqrt[\mathbb{R}]{\operatorname{supp} Q}$. We also have $\operatorname{supp} \widetilde{Q} = \sqrt[\mathbb{R}]{\operatorname{supp} Q}$ so that if $\operatorname{supp} Q$ is not real radical then $Q \subsetneq \widetilde{Q}$. Example 3.3.14 is such a case where $\operatorname{supp} Q \neq \sqrt[\mathbb{R}]{\operatorname{supp} Q}$. We notice that, by Theorem 3.4.3 and [Sch05b, th. 3.17], if Q is *stable* (see Definition 1.1.27) then $\widetilde{Q} = \overline{Q}$. But the inclusion $\widetilde{Q} = Q + \sqrt{\operatorname{supp} Q} \subset \overline{Q}$ can be strict, as shown by the following example.

Example 3.4.4 ([Sch05a, ex. 3.2], [Sch05b, rem. 3.15], Example 3.3.12)**.** Let $Q = \mathcal{Q}(1-x^2-y^2, -xy, x-y, y-x^2) \subset \mathbb{R}[x,y]$. Notice that $S = \mathcal{S}(Q) = \{\mathbf{0}\}$ and that Q is Archimedean. Therefore, by Theorem 1.1.22, $\overline{Q} = \operatorname{Pos}(\{\mathbf{0}\})$ (see also Section 1.3.5). We verify that $\operatorname{supp} Q = (0)$ and that $\mathcal{I}(S) = \operatorname{supp}\overline{Q} = (x,y)$. Since $\operatorname{supp} Q \neq \operatorname{supp}\overline{Q}$, finally we have $Q = Q + \sqrt{\operatorname{supp} Q} = \widetilde{Q} \subsetneq \overline{Q}$.

Theorem 3.4.3 suggests the idea that, when we consider the moment hierarchy, we are extending the quadratic module $\mathcal{Q}(\mathbf{g})$ to $\mathcal{Q}(\mathbf{g}, \pm\mathbf{h})$, where $\mathbf{h}$ are generators of $\sqrt[\mathbb{R}]{\operatorname{supp}\mathcal{Q}(\mathbf{g})}$. We specify this idea in Lemma 3.4.5, Proposition 3.4.8 and Theorem 3.4.11, investigating the relations between the truncated parts of $\mathcal{L}_d(\mathbf{g})$.

Lemma 3.4.5. *Let* $J = \sqrt[\mathbb{R}]{\operatorname{supp}\mathcal{Q}(\mathbf{g})}$. *If* $(\mathbf{h}) \subset J$, $\deg\mathbf{h} \leq t$, *then* $\exists d \geq t$ *such that* $\langle\mathbf{h}\rangle_t \subset \overline{\mathcal{Q}_d(\mathbf{g})}$. *In this case:*

$$\mathcal{L}_d(\mathbf{g})^{[t]} \subset \mathcal{L}_t(\mathbf{g}, \pm\mathbf{h}) \subset \mathcal{L}_t(\mathbf{g}),$$

and in particular $\mathcal{L}_d(\mathbf{g})^{[t]} \subset \mathcal{L}_t(\pm\mathbf{h})$. *Moreover,* $\mathcal{L}_{d+2k}(\mathbf{g})^{[t+k]} \subset \mathcal{L}_{t+k}(\pm\mathbf{h})$ *for all* $k \in \mathbb{N}$.

Proof. By Lemma 3.4.2, $\langle\mathbf{h}\rangle_t \subset (\mathbf{h})_t \subset \overline{\mathcal{Q}_d(\mathbf{g})}$ for some $d \geq t$. Let $h \in \mathbf{h}$ and $f \in \mathbb{R}[\mathbf{x}]_{t-\deg h}$. Then $\pm fh \in \overline{\mathcal{Q}_d(\mathbf{g})}$, and for $\Lambda \in \mathcal{L}_d(\mathbf{g})$, we have $\left\langle \Lambda^{[t]} \middle| fh \right\rangle = \langle \Lambda | fh \rangle = 0$, i.e. $\mathcal{L}_d(\mathbf{g})^{[t]} \subset \mathcal{L}_t(\mathbf{g}, \pm\mathbf{h})$. The other inclusion $\mathcal{L}_t(\mathbf{g}, \pm\mathbf{h}) \subset \mathcal{L}_t(\mathbf{g})$ follows by definition.

For the second part, notice that $\langle\mathbf{h}\rangle_{t+k} \subset \overline{\mathcal{Q}_{d+2k}(\mathbf{g})}$. Indeed, if $p \in \langle\mathbf{h}\rangle_{t+k}$ then $p = \sum_i \mathbf{x}^{\alpha(i)} p_i$, where $p_i \in \langle\mathbf{h}\rangle_t \subset \overline{\mathcal{Q}_d(\mathbf{g})}$ and $|\alpha(i)| \leq k$. Writing $\mathbf{x}^{\alpha(i)} = (\frac{\mathbf{x}^{\alpha(i)}+1}{2})^2 - (\frac{\mathbf{x}^{\alpha(i)}-1}{2})^2$, we deduce that $p = \sum_i (\frac{\mathbf{x}^{\alpha(i)}+1}{2})^2 p_i + (\frac{\mathbf{x}^{\alpha(i)}-1}{2})^2(-p_i) \in \overline{\mathcal{Q}_{d+2k}(\mathbf{g})}$, i.e. $\langle\mathbf{h}\rangle_{t+k} \subset \overline{\mathcal{Q}_{d+2k}(\mathbf{g})}$. Then we can conclude the proof as in the first part. □

Lemma 3.4.5 says that the moment hierarchy $(\mathcal{L}_{2d}(\mathbf{g}))_{d\in\mathbb{N}}$ is equivalent to the moment hierarchy $(\mathcal{L}_{2d}(\mathbf{g}, \pm\mathbf{h}))_{d\in\mathbb{N}}$, where $(\mathbf{h}) = \sqrt[\mathbb{R}]{\operatorname{supp}\mathcal{Q}(\mathbf{g})}$. Lemma 3.4.5 is an algebraic result, in the sense that $\operatorname{supp}\mathcal{Q}(\mathbf{g})$ may be unrelated to the geometry $\mathcal{S}(\mathbf{g})$ that $\mathbf{g}$ defines. If some additional conditions hold (namely if we have only equalities, or a preordering, or a small dimension), it can however provide geometric characterizations.

Corollary 3.4.6. *Suppose that* $\mathcal{S}(\mathbf{g}) \subset \mathcal{V}_{\mathbb{R}}(\mathbf{h})$. *Then for every* $t_0 \geq \deg\mathbf{h}$ *there exists* $t_1 \geq t_0$ *such that:*

$$\mathcal{L}_{t_1}(\Pi\mathbf{g})^{[t_0]} \subset \mathcal{L}_{t_0}(\pm\mathbf{h}).$$

In particular this holds when $(\mathbf{h}) = \mathcal{I}(\mathcal{S}(\mathbf{g}))$.

Moreover, $\mathcal{L}_{t_1+2k}(\mathbf{g})^{[t_0+k]} \subset \mathcal{L}_{t_0+k}(\pm\mathbf{h})$ *for all* $k \in \mathbb{N}$.

Proof. $\mathcal{S}(\mathbf{g}) \subset \mathcal{V}_{\mathbb{R}}(\mathbf{h})$ if and only if $\sqrt[\mathbb{R}]{(\mathbf{h})} \subset \mathcal{I}(\mathcal{S}(\mathbf{g})) = \sqrt[\mathbb{R}]{\operatorname{supp}\mathcal{Q}(\Pi\mathbf{g})}$ by the Real Nullstellensatz, Theorem 1.1.25. Then we can apply Lemma 3.4.5. □

Corollary 3.4.7. *Let* $Q = \mathcal{Q}(\mathbf{g})$. *Suppose that* $\mathcal{S}(\mathbf{g}) \subset \mathcal{V}_{\mathbb{R}}(\mathbf{h})$ *and* $\dim \frac{\mathbb{R}[\mathbf{x}]}{\operatorname{supp} Q} \leq 1$. *Then for every* $t_0 \geq \deg\mathbf{h}$ *there exists* $t_1 \geq t_0$ *such that* $(\mathbf{h})_{t_0} \subset \overline{\mathcal{Q}_{t_1}(\mathbf{g})}$. *In this case:*

$$\mathcal{L}_{t_1}(\mathbf{g})^{[t_0]} \subset \mathcal{L}_{t_0}(\pm\mathbf{h}),$$

ans in particular this holds when $(\mathbf{h}) = \mathcal{I}(\mathcal{S}(\mathbf{g}))$.

Moreover, $\mathcal{L}_{t_1+2k}(\mathbf{g})^{[t_0+k]} \subset \mathcal{L}_{t_0+k}(\pm\mathbf{h})$ *for all* $k \in \mathbb{N}$.

Proof. We prove it as Corollary 3.4.6, using Theorem 1.1.26 instead of the Real Nullstellensatz. □

We mention now a strong duality result, that is useful to produce examples and counterexamples for exactness and finite convergence. This result, very similar to a result in [Mar03], generalizes the condition $\operatorname{supp} Q = 0$ in Theorem 3.3.7.

Proposition 3.4.8. *Let $Q = \mathcal{Q}(\mathbf{g})$ be a finitely generated quadratic module, and let $\mathbf{h}$ be a graded basis of $\sqrt[\mathbb{R}]{\operatorname{supp} Q}$. Then for any d we have $\mathcal{Q}_d(\mathbf{g}, \pm\mathbf{h}) = \overline{\mathcal{Q}_d(\mathbf{g}, \pm\mathbf{h})}$ is closed. Moreover, if we consider the extended hierarchies $\mathcal{Q}_{2d}(\mathbf{g}, \pm\mathbf{h})$ and $\mathcal{L}_{2d}(\mathbf{g}, \pm\mathbf{h})$, then for any $f \in \mathbb{R}[\mathbf{x}]$ such that $f^* > -\infty$ we have that $f^*_{\mathrm{SoS},d}$ is attained (i.e. $f - f^*_{\mathrm{SoS},d} \in \mathcal{Q}_{2d}(\mathbf{g}, \pm\mathbf{h})$) and there is no duality gap: $f^*_{\mathrm{SoS},d} = f^*_{\mathrm{Mom},d}$.*

Proof. By [Mar08, lem. 4.1.4], $\mathcal{Q}_{2d}(\mathbf{g}, \pm\mathbf{h}) = \mathcal{Q}_{2d}(\mathbf{g}) + I_{2d}$ is closed. Therefore we have $\mathcal{L}_{2d}(\mathbf{g}, \pm\mathbf{h})^\vee = (\mathcal{Q}_{2d}(\mathbf{g}, \pm\mathbf{h}))^{\vee\vee} = \overline{\mathcal{Q}_{2d}(\mathbf{g}, \pm\mathbf{h})} = \mathcal{Q}_{2d}(\mathbf{g}, \pm\mathbf{h})$, from which we deduce that there is not duality gap, by classical convexity arguments, as follows.

Let $m > f^*_{\mathrm{SoS},d}$, so that $f - m \notin \mathcal{Q}_{2d}(\mathbf{g}, \pm\mathbf{h}) = \overline{\mathcal{Q}_{2d}(\mathbf{g}, \pm\mathbf{h})}$. From the separation theorem (Theorem 1.2.1) there exists $\Lambda \in \mathcal{Q}_{2d}(\mathbf{g}, \pm\mathbf{h})^\vee = \mathcal{L}_{2d}(\mathbf{g}, \pm\mathbf{h})$ such that $\langle \Lambda | f - m \rangle < 0$. Proceeding as in Example 3.2.1, we can assume that $\langle \Lambda | 1 \rangle = 1$, and thus $\Lambda \in \mathcal{L}^{(1)}_{2d}(\mathbf{g}, \pm\mathbf{h})$. This shows that $\langle \Lambda | f \rangle \leq m$ and $f^*_{\mathrm{Mom},d} \leq f^*_{\mathrm{SoS},d}$. Since the other inequality is always satisfied, we have proven $f^*_{\mathrm{Mom},d} = f^*_{\mathrm{SoS},d}$, i.e. strong duality. □

We conjecture that, more generally, there is no duality gap when Q is reduced (i.e. $\operatorname{supp} Q = \sqrt[\mathbb{R}]{\operatorname{supp} Q}$) without adding the generators of the radical of the support.

3.4.2 Annihilators of truncated moment sequences

Recall that the annihilator $\operatorname{Ann}_t(\Lambda)$ is the kernel of the moment matrix of Λ (or of the Hankel operator), see Section 1.2.1 and Section 1.2.2. With the characterization of $\widetilde{Q}$, Theorem 3.4.3, we can now describe these kernels of moment matrices associated to truncated positive linear functionals.

We recall the definition of genericity in the truncated setting and equivalent characterizations.

Definition 3.4.9. Let $C \subset \mathcal{L}_{2d}(\mathbf{g})$ be a convex set. We say that $\Lambda^* \in C$ is *generic* in C if $\operatorname{rank} H^d_{\Lambda^*} = \max\{\operatorname{rank} H^d_\eta \mid \eta \in C\}$.

In particular, we will consider generic Λ^* in the following convex sets:

- $C = \mathcal{L}_{2d}(\mathbf{g})$, the cone positive linear functionals, or the feasible pseudo-moment sequences of Lasserre's moment relaxation of order d;
- $C = \mathcal{L}^{(1)}_{2d}(\mathbf{g})$, the convex set defined as the section of $\mathcal{L}_{2d}(\mathbf{g})$ given by $\langle \Lambda | 1 \rangle = 1$;
- $C = \mathcal{L}^{\min}_{2d}(\mathbf{g})$, the exposed face of $\mathcal{L}^{(1)}_{2d}(\mathbf{g})$ defined by $\langle \Lambda | f \rangle = f^*_{\mathrm{Mom},d}$.
- $C = \mathcal{L}_{2d}(\mathbf{g})^{[2k]}, \mathcal{L}^{(1)}_{2d}(\mathbf{g})^{[2k]}, \mathcal{L}^{\min}_{2d}(\mathbf{g})^{[2k]}$, the restriction of the positive linear functionals to $\mathbb{R}[\mathbf{x}]^*_{2k}$ (or the truncation of the pseudo-moment sequences to degree $\leq 2k$). .

This genericity can be characterized as follows, see [Las+13, prop. 4.7].

Proposition 3.4.10. *Let $\Lambda \in C \subset \mathcal{L}_{2d}(\mathbf{g})$. The following are equivalent:*

(i) Λ *is generic in* C*;*

(ii) $\mathrm{Ann}_d(\Lambda) \subset \mathrm{Ann}_d(\eta)\ \forall \eta \in C$*;*

(iii) $\forall k \le d$*, we have:* $\mathrm{rank}\, H_\Lambda^k = \max\{\mathrm{rank}\, H_\eta^k \mid \eta \in C\}$.

Proof. We start proving that $(i) \Rightarrow (ii)$. Let $\Lambda \in C \subset \mathcal{L}_{2d}(\mathbf{g})$ be generic, and let $\eta \in C$. Since C is convex, then $\frac{1}{2}(\Lambda + \eta) \in C$ and $\mathrm{Ann}_d(\frac{1}{2}(\Lambda+\eta)) = \mathrm{Ann}_d(\Lambda) \cap \mathrm{Ann}_d(\eta)$. Indeed, the inclusion

$$\mathrm{Ann}_d(\frac{1}{2}(\Lambda+\eta)) = \ker H^d_{\frac{1}{2}(\Lambda+\eta)} = \ker H^d_{\Lambda+\eta} \supset \ker H^d_\Lambda \cap \ker H^d_\eta = \mathrm{Ann}_d(\Lambda) \cap \mathrm{Ann}_d(\eta)$$

is obvious. Conversely, if $f \in \mathrm{Ann}_d(\frac{1}{2}(\Lambda+\eta))$ then $f \star (\frac{1}{2}(\Lambda+\eta)) = \frac{1}{2}(f \star \Lambda + f \star \eta) = 0$. In particular,

$$0 = \left\langle f \star (\frac{1}{2}(\Lambda+\eta)) \middle| f \right\rangle = \frac{1}{2}\left(\langle f \star \Lambda | f\rangle + \langle f \star \eta | f \rangle\right) = \frac{1}{2}\left(\left\langle \Lambda \middle| f^2 \right\rangle + \left\langle \eta \middle| f^2 \right\rangle\right)$$

Therefore $\left\langle \Lambda \middle| f^2 \right\rangle = \left\langle \eta \middle| f^2 \right\rangle = 0$, and from Lemma 1.3.10 we have $f \in \mathrm{Ann}_d(\Lambda) \cap \mathrm{Ann}_d(\eta)$, proving the reverse inclusion. Therefore, $\mathrm{Ann}_d(\frac{1}{2}(\Lambda+\eta)) = \mathrm{Ann}_d(\Lambda) \cap \mathrm{Ann}_d(\eta) \subset \mathrm{Ann}_d(\Lambda)$. Now, since Λ is generic, $\mathrm{rank}\, H^d_\Lambda \ge \mathrm{rank}\, H^d_{\frac{1}{2}(\Lambda+\eta)}$. Then $\dim \mathrm{Ann}_d(\frac{1}{2}(\Lambda+\eta)) \le \dim \mathrm{Ann}_d(\Lambda)$, and we have $\mathrm{Ann}_d(\Lambda) \cap \mathrm{Ann}_d(\eta) = \mathrm{Ann}_d(\frac{1}{2}(\Lambda+\eta)) = \mathrm{Ann}_d(\Lambda)$. This shows that $\mathrm{Ann}_d(\Lambda) \subset \mathrm{Ann}_d(\eta)$, concluding the proof.

We prove now $(ii) \Rightarrow (iii)$. We first show that, if $k \le d$, $\mathrm{Ann}_k(\Lambda) = \mathrm{Ann}_d(\Lambda) \cap \mathbb{R}[\mathbf{x}]_k$. Indeed, if $f \in \mathrm{Ann}_d(\Lambda) \cap \mathbb{R}[\mathbf{x}]_k$ then $(f \star \Lambda)^{[d]} = 0$, and thus $(f \star \Lambda)^{[k]} = 0$, i.e. $f \in \mathrm{Ann}_k(\Lambda)$. On the contrary, if $f \in \mathrm{Ann}_k(\Lambda)$ then $\left\langle \Lambda^{[2k]} \middle| f^2 \right\rangle = \left\langle \Lambda^{[2d]} \middle| f^2 \right\rangle = 0$, and thus $f \in \mathrm{Ann}_d(\Lambda)$ from Lemma 1.3.10. Therefore:

$$\ker H^k_\Lambda = \mathrm{Ann}_k(\Lambda) = \mathrm{Ann}_d(\Lambda) \cap \mathbb{R}[\mathbf{x}]_k \subset \mathrm{Ann}_d(\eta) \cap \mathbb{R}[\mathbf{x}]_k = \mathrm{Ann}_k(\eta) = \ker H^k_\eta$$

since Λ satisfies (ii). Finally, the inclusion $\ker H^k_\Lambda \subset \ker H^k_\eta$ implies $\mathrm{rank}\, H^k_\Lambda \ge \mathrm{rank}\, H^k_\eta$ for all $k \le d$ and all $\eta \in C$.

The last implication $(iii) \Rightarrow (i)$ is obvious. □

Remark. By Proposition 3.4.10 notice that $\forall k \le d$, if $\Lambda^* \in C \subset \mathcal{L}_{2d}(\mathbf{g})$ is generic in C then $(\Lambda^*)^{[2k]}$ is generic in $C^{[2k]}$. In particular, $\mathrm{Ann}_k(\Lambda^*) \subset \mathrm{Ann}_k(\eta)\ \forall \eta \in C$.

We can show that linear functionals in the relative interior of $C \subset \mathcal{L}_{2d}(\mathbf{g})$ are generic, using the ideas of the proof above. Indeed, let $\Lambda = \frac{1}{2}(\Lambda_1 + \Lambda_2)$ be a convex combination of elements of C. The inclusion

$$\mathrm{Ann}_d(\Lambda) = \ker H^d_\Lambda = \ker H^d_{\Lambda_1+\Lambda_2} \supset \ker H^d_{\Lambda_1} \cap \ker H^d_{\Lambda_2} = \mathrm{Ann}_d(\Lambda_1) \cap \mathrm{Ann}_d(\Lambda_2)$$

is obvious. Conversely, if $f \in \mathrm{Ann}_d(\Lambda)$ then $f \star \Lambda = \frac{1}{2}(f \star \Lambda_1 + f \star \Lambda_2) = 0$. In particular,

$$0 = \langle f \star \Lambda | f \rangle = \frac{1}{2}(\langle f \star \Lambda | f\rangle + \langle f \star \Lambda | f \rangle) = \frac{1}{2}(\left\langle \Lambda_1 \middle| f^2 \right\rangle + \left\langle \Lambda_2 \middle| f^2 \right\rangle)$$

Therefore $\left\langle \Lambda_1 \middle| f^2 \right\rangle = \left\langle \Lambda_2 \middle| f^2 \right\rangle = 0$, and from Lemma 1.3.10 we have $f \in \mathrm{Ann}_d(\Lambda_1) \cap \mathrm{Ann}_d(\Lambda_2)$, proving the reverse inclusion. Notice that there are examples where extremal points are generic, see Example 3.5.2.

If we use an SDP solver based on interior point method to solve a Lasserre's moment relaxation, we will (approximately) get a pseudo-moment sequence in the relative interior of the exposed face $\mathcal{L}_{2d}^{\min} = \mathcal{L}_{2d}^{(1)}(\mathbf{g}) \cap \{\langle \Lambda | f \rangle = f^*_{\mathrm{Mom},d}\}$, which is then generic in this face. We will use generic linear functionals to recover the minimizers when we have exactness or the flat truncation property.

We are now ready to describe the annihilator of generic elements.

Theorem 3.4.11. *Let* $Q = \mathcal{Q}(\mathbf{g})$ *and* $J = \sqrt[\mathbb{R}]{\operatorname{supp} Q}$. *Then for all* $d, t \in \mathbb{N}$ *big enough and for* $\Lambda^* \in \mathcal{L}_d(\mathbf{g})$ *generic, we have* $J = (\mathrm{Ann}_t(\Lambda^*))$. *Moreover if* $Q = O$ *is a preordering, then* $(\mathrm{Ann}_t(\Lambda^*)) = I(\mathcal{S}(\mathbf{g}))$.

Proof. Let $t \in \mathbb{N}$ such that J is generated in degree $\leq t$, by the graded basis $\mathbf{h} = h_1, \dots, h_s$. From Lemma 3.4.2 we deduce that there exists $d \in \mathbb{N}$ such that $J_{2t} \subset \overline{\mathcal{Q}_d(\mathbf{g})}$. Let $\Lambda^* \in \mathcal{L}_d(\mathbf{g})$ generic.

We first prove that $J \subset (\mathrm{Ann}_t(\Lambda^*))$. By Proposition 3.4.10 we have $\mathrm{Ann}_t(\Lambda^*) = \bigcap_{\Lambda \in \mathcal{L}_d(\mathbf{g})} \mathrm{Ann}_t(\Lambda)$. Then it is enough to prove that $J_t \subset \mathrm{Ann}_t(\Lambda)$ for all $\Lambda \in \mathcal{L}_d(\mathbf{g})$.

By Lemma 3.4.5 $\mathcal{L}_d(\mathbf{g})^{[2t]} \subset \mathcal{L}_{2t}(\pm \mathbf{h}) \subset \langle \mathbf{h} \rangle_{2t}^{\perp}$. Then $\forall f \in J_t = \langle \mathbf{h} \rangle_t$, $\forall p \in \mathbb{R}[\mathbf{x}]_t$, $\forall \Lambda \in \mathcal{L}_d(\mathbf{g})$, we have $f p \in \langle \mathbf{h} \rangle_{2t}$ and $\left\langle \Lambda^{[2t]} \middle| f p \right\rangle = 0$. This shows that $H_\Lambda^t(f)(p) = \left\langle (f \star \Lambda)^{[t]} \middle| p \right\rangle = \langle \Lambda | f p \rangle = 0$, i.e. $f \in \mathrm{Ann}_t(\Lambda) = \ker H_\Lambda^t$.

Conversely, we show that $(\mathrm{Ann}_t(\Lambda^*)) \subset J$ for Λ^* generic in $\mathcal{L}_d(\mathbf{g})$. Since $J = \operatorname{supp} \widetilde{Q} = \operatorname{supp} \overline{\bigcup_j \mathcal{Q}_j(\mathbf{g})}$ (by Theorem 3.4.3) it is enough to prove that $\mathrm{Ann}_t(\Lambda^*) \subset \overline{\mathcal{Q}_d(\mathbf{g})} \cap -\overline{\mathcal{Q}_d(\mathbf{g})} = \operatorname{supp} \overline{\mathcal{Q}_d(\mathbf{g})} = \operatorname{supp} \mathcal{L}_d(\mathbf{g})^\vee$.

Let $f \in \mathrm{Ann}_t(\Lambda^*) = \bigcap_{\Lambda \in \mathcal{L}_k(\mathbf{g})} \mathrm{Ann}_t(\Lambda)$ (we use again Proposition 3.4.10) and let $\Lambda \in \mathcal{L}_d(\mathbf{g})$. Then $\langle \Lambda | f \rangle = \left\langle (f \star \Lambda)^{[t]} \middle| 1 \right\rangle = H_\Lambda^t(f)(1) = 0$. In particular $f \in \mathcal{L}_d(\mathbf{g})^\vee$. We prove that $-f \in \mathcal{L}_d(\mathbf{g})^\vee$ in the same way. Then $f \in \mathcal{L}_d(\mathbf{g})^\vee \cap -\mathcal{L}_d(\mathbf{g})^\vee = \overline{\mathcal{Q}_d(\mathbf{g})} \cap -\overline{\mathcal{Q}_d(\mathbf{g})} = \operatorname{supp} \overline{\mathcal{Q}_d(\mathbf{g})}$, and finally we deduce from Definition 3.4.9 and Theorem 3.4.3 that $\mathrm{Ann}_t(\Lambda^*) \subset \operatorname{supp} \widetilde{Q} = J$.

The second part follows from the first one and the Real Nullstelensatz, Theorem 1.1.25. □

Theorem 3.4.11 shows the possibilities and the limits of moment hierarchies. For instance, we cannot expect exactness of the moment hierarchy $\mathcal{L}_{2d}(\mathbf{g})$ for any objective function f (or in other words $\mathcal{L}_{2d}(\mathbf{g})^{[k]} \subset \mathcal{M}(S)^{[k]}$) if $\sqrt[\mathbb{R}]{\operatorname{supp} Q} \neq \mathcal{I}(S)$: see Example 3.3.12.

In Proposition 3.4.12 we investigate the infinite dimensional case. We say that $\Lambda^* \in \mathcal{L}(Q) = Q^\vee$ is *generic* if $\mathrm{Ann}(\Lambda^*) \subset \mathrm{Ann}(\Lambda)$ for all $\Lambda \in \mathcal{L}(Q)$, using Proposition 3.4.10 to have the analogy with the finite dimensional case.

Proposition 3.4.12. *Let* Q *be a quadratic module,* $S = \mathcal{S}(Q)$ *and* $\Lambda^* \in \mathcal{L}(Q) = Q^\vee$ *be generic. Then* $\sqrt[\mathbb{R}]{\operatorname{supp} Q} \subset \mathrm{Ann}(\Lambda^*) \subset \mathcal{I}(S)$. *Moreover:*

(i) if Q *is Archimedean then* $\mathrm{Ann}(\Lambda^*) = \mathcal{I}(S)$;

(ii) if $Q = O$ *is a preordering,* $\mathrm{Ann}(\Lambda^*) = \mathcal{I}(S)$;

(iii) if I *is an ideal of* $\mathbb{R}[\mathbf{x}]$ *and* $\Lambda^* \in \mathcal{L}(I) = (I + \Sigma^2)^\vee$ *is generic, then* $\mathrm{Ann}(\Lambda^*) = \sqrt[\mathbb{R}]{I}$.

Proof. For $x \in S$, notice that $\mathbf{e}_x \in \mathcal{L}(Q)$. Then, since Λ^* is generic:

$$\mathrm{Ann}(\Lambda^*) \subset \bigcap_{x \in S} \mathrm{Ann}(\mathbf{e}_x) = \bigcap_{x \in S} \mathcal{I}(x) = \mathcal{I}(S).$$

Now observe that $\operatorname{supp} Q \subset \operatorname{Ann}(\Lambda^*)$ by definition. Since $\operatorname{Ann}(\Lambda^*)$ is a real radical ideal (see [Las+13, prop. 3.13]), then $\sqrt[\mathbb{R}]{\operatorname{supp} Q} \subset \operatorname{Ann}(\Lambda^*)$.

If Q is Archimedean, then by Putinar's Positivstellensatz (Theorem 1.3.7) $\mathcal{L}(Q) = Q^\vee = \mathcal{M}(S)$. In particular Λ^* is a measure $\mu \in \mathcal{M}(S)$ supported on S: $\forall f \in \mathbb{R}[\mathbf{x}]$, $\langle \Lambda^* | f \rangle = \int f \, \mathrm{d}\mu$. Let $h \in \mathcal{I}(S)$ and $f \in \mathbb{R}[\mathbf{x}]$. Then:

$$\langle \Lambda^* | f h \rangle = \int f h \, \mathrm{d}\mu = \int 0 \, \mathrm{d}\mu = 0,$$

i.e. $h \in \operatorname{Ann}(\Lambda^*)$, which proves the reverse inclusion.

If $Q = O$ is a preordering then $\sqrt[\mathbb{R}]{\operatorname{supp} O} = \mathcal{I}(S)$ by the Real Nullstellensatz, Theorem 1.1.25. Then $\operatorname{Ann}(\Lambda^*) = \mathcal{I}(S)$.

As $\mathcal{L}(I) = \mathcal{L}(I + \Sigma^2)$, the last point follows from the previous one applied to $O = I + \Sigma^2$. □

If we compare Theorem 3.4.11 and Proposition 3.4.12, we see that the description in the infinite dimensional setting is more complicated, as we don't always have the equality $\operatorname{Ann}(\Lambda^*) = \sqrt[\mathbb{R}]{\operatorname{supp} Q}$, see the case of an Archimedean quadratic module. This happens because limit properties that appear in the infinite dimensional case do not show up in the truncated setting.

3.4.3 Regularity, moment sequences and flat truncation

In this section, we analyze the properties of moment sequences in $\mathcal{L}_d(\mathbf{g})$ when $S = \mathcal{S}(\mathbf{g})$ is finite. We will use the results in this section to study the case of finitely many minimizers in Polynomial Optimization problems, and in particular flat truncation.

We start briefly recalling the definition of *Castelnuovo-Mumford regularity*, and its relations with the *interpolation degree* in the zero dimensional case. We refer to [Eis05] for more details.

Let $\Xi = \{\xi_1, \ldots, \xi_r\} \subset \mathbb{C}^n$ be a finite set of (complex) points and let $\mathcal{I}(\Xi) = \{p \in \mathbb{C}[X] \mid p(\xi_i) = 0 \ \forall i \in 1, \ldots, r\}$ be the complex vanishing ideal of the points Ξ. The *Castelnuovo-Mumford regularity* of an ideal I (resp. Ξ) is $\max_i(\deg S_i - i)$ where S_i is the i^{th} module of syzygies in a minimal resolution of I (resp. $\mathcal{I}(\Xi)$). Let denote it by $\rho(I)$ (resp. $\rho(\Xi)$).

It is well known that Ξ admits a family of interpolator polynomials $(u_i) \subset \mathbb{C}[\mathbf{X}]$ such that $u_i(\xi_j) = \delta_{i,j}$. The minimal degree $\iota(\Xi)$ of a family of interpolator polynomials is called the *interpolation degree* of Ξ. Since a family of interpolator polynomials (p_i) is a basis of $\mathbb{C}[\mathbf{X}]/\mathcal{I}(\Xi)$, the ideal $\mathcal{I}(\Xi)$ is generated in degree $\leq \iota(\Xi) + 1$ and $\rho(\Xi) \leq \iota(\Xi) + 1$. A classical result [Eis05, th. 4.1] relates the interpolation degree of Ξ with its regularity, and the minimal degree of a basis of $\mathbb{C}[\mathbf{X}]/\mathcal{I}(\Xi)$. This result can be stated as follows, for real points $\Xi \subset \mathbb{R}^n$:

Proposition 3.4.13. *Let $\Xi = \{\xi_1, \ldots, \xi_r\} \subset \mathbb{R}^r$ with regularity $\rho(\Xi)$. Then the minimal degree $\iota(\Xi)$ of a basis of $\mathbb{R}[\mathbf{X}]/\mathcal{I}(\Xi)$ is $\rho(\Xi) - 1$ and there exist interpolator polynomials $u_1, \ldots, u_r \in \mathbb{R}[\mathbf{X}]_{\rho(\Xi)-1}$.*

Another property that we will use is the following:

Proposition 3.4.14 ([BS87])**.** *Any ideal $I \subset \mathbb{R}[\mathbf{X}]$ has a graded basis in degree less than or equal to its regularity $\rho(I)$.*

In particular, for a set of points $\Xi = \{\xi_1, \ldots, \xi_r\}$, the ideal $\mathcal{I}(\Xi)$ has a graded basis of degree equal to the regularity $\rho(\Xi)$, that can be computed as a Groebner basis with respect to a monomial ordering refining the degree, see Section 1.1.3. The minimal degree of a monomial basis B of $\mathbb{R}[\mathbf{X}]/\mathcal{I}(\Xi)$ is $\iota(\Xi) = \rho(\Xi) - 1$. Such a finite basis B can be chosen so that it is stable by monomial division.

We now turn back our attention to truncated positive linear functional, in particular in the zero dimensional case. The next result shows that truncated positive linear functionals (or pseudo-moment sequences) orthogonal to the vanishing ideal of the points, truncated above twice the regularity are coming from measures.

Proposition 3.4.15. *Let $\Xi = \{\xi_1,\ldots,\xi_r\} \subset \mathbb{R}^n$, $I = \mathcal{I}(\Xi)$ its real vanishing ideal and let $\rho = \rho(\Xi)$ the regularity of Ξ. For $t \geq \rho - 1$, $\Lambda \in I_t^{\perp}$ if and only if $\Lambda \in \langle \mathbf{e}_{\xi_1}^{[t]},\ldots,\mathbf{e}_{\xi_r}^{[t]}\rangle$. Moreover if $t \geq \rho - 1$ and $\Lambda \in \mathcal{L}_{2t}(I_{2t})$, then $\Lambda \in \operatorname{cone}(\mathbf{e}_{\xi_1}^{[2t]},\ldots,\mathbf{e}_{\xi_r}^{[2t]})$ and $\operatorname{rank} H_\Lambda^t = r$.*

Proof. Let $u_1,\ldots,u_r \in \mathbb{R}[\mathbf{x}]_t$ be interpolation polynomials of degree $\leq \rho - 1 \leq t$ (Proposition 3.4.13). Consider the sequence of vector space maps:

$$\begin{array}{ccc} 0 \to I_t \to \mathbb{R}[\mathbf{x}]_t & \overset{\psi}{\longrightarrow} & \langle u_1,\ldots,u_r\rangle \to 0 \\ p & \mapsto & \sum_{i=1}^{r} p(\xi_i) u_i, \end{array}$$

which is exact since $\ker\psi = \{p \in \mathbb{R}[\mathbf{x}]_t \mid p(\xi_i) = 0\} = I_t$. Therefore we have $\mathbb{R}[\mathbf{x}]_t = \langle u_1,\ldots,u_r\rangle \oplus I_t$.

Let $\Lambda \in I_t^{\perp}$. Then $\tilde{\Lambda} = \Lambda - \sum_{i=1}^r \langle \Lambda | u_i\rangle \mathbf{e}_{\xi_i}^{[t]} \in I_t^{\perp}$ is such that $\left\langle \tilde{\Lambda} \middle| u_i \right\rangle = 0$ for $i = 1,\ldots,r$. Thus, $\tilde{\Lambda} \in \langle u_1,\ldots,u_r\rangle^{\perp} \cap I_t^{\perp} = (\langle u_1,\ldots,u_r\rangle \oplus I_t)^{\perp} = \mathbb{R}[\mathbf{x}]_t^{\perp}$, i.e. $\tilde{\Lambda} = 0$ showing that $I_t^{\perp} \subset \langle \mathbf{e}_{\xi_1}^{[t]},\ldots,\mathbf{e}_{\xi_r}^{[t]}\rangle$. The reverse inclusion is direct since I_t is the space of polynomials of degree $\leq t$ vanishing at ξ_i for $i = 1,\ldots,r$.

Assume that $t \geq \rho - 1$ and $\Lambda \in \mathcal{L}_{2t}(I_{2t})$. Then $\Lambda \in I_{2t}^{\perp}$ and $\left\langle \Lambda \middle| p^2 \right\rangle \geq 0$ for any $p \in \mathbb{R}[\mathbf{x}]_t$. By the previous analysis,

$$\Lambda = \sum_{i=1}^{r} \omega_i \mathbf{e}_{\xi_i}^{[2t]}$$

As $0 \leq \left\langle \Lambda \middle| u_i^2 \right\rangle = \omega_i$ for $i = 1,\ldots,r$, we deduce that $\Lambda \in \operatorname{cone}(\mathbf{e}_{\xi_1}^{[t]},\ldots,\mathbf{e}_{\xi_r}^{[t]})$.

We verify that the image of $H_\Lambda^t : p \in \mathbb{R}[\mathbf{x}]_t \mapsto \sum_{i=1}^r \omega_i p(\zeta_i) \mathbf{e}_{\xi_i}^{[t]}$ is $\langle \mathbf{e}_{\xi_1}^{[t]},\ldots,\mathbf{e}_{\xi_r}^{[t]}\rangle$, computing $H_\Lambda^t(u_i)$ for $i = 1,\ldots,r$. Thus, $\operatorname{rank} H_\Lambda^t = \dim\langle \mathbf{e}_{\xi_1}^{[t]},\ldots,\mathbf{e}_{\xi_r}^{[t]}\rangle = r$ since $(\mathbf{e}_{\xi_i}^{[t]})_{i=1,\ldots,r}$ is the dual basis of $(u_i)_{i=1,\ldots,r}$: $\left\langle \mathbf{e}_{\xi_i}^{[t]} \middle| u_j \right\rangle = u_j(\xi_i) = \delta_{ij}$. □

We deduce another corollary, giving degree bounds for the case of a graded basis of a real radical ideal.

Proposition 3.4.16. *Let $I = \sqrt[\mathbb{R}]{I}$ be a real radical ideal and $\mathbf{h} = h_1,\ldots,h_m$ be a graded basis of I. Then for all $d \geq \deg\mathbf{h} = \max_i(\deg h_i)$ and $\Lambda^* \in \mathcal{L}_{2d}(\pm\mathbf{h})$ generic, we have $\operatorname{Ann}_d(\Lambda^*) = I_d$.*

Proof. Let $d \geq \max_i(\deg h_i)$ and $\Lambda^* \in \mathcal{L}_{2d}(\pm\mathbf{h})$ generic. Then for all i we have $(h_i\star\Lambda^*)^{[d-\deg h_i]} = 0$. Now let $p \in I_d$. Since h is a graded basis we have $p = \sum_{i=1}^m p_i h_i$, where $\deg p_i \leq d - \deg h_i$. Notice that $(p\star\Lambda^*)^{[d]} = \sum_{i=1}^m (p_i\star(h_i\star\Lambda^*))^{[d]}$. As $(h_i\star\Lambda^*)^{[d-\deg h_i]} = 0$ we conclude $(p\star\Lambda^*)^{[d]} = 0$ and thus $p \in \operatorname{Ann}_d(\Lambda^*)$. Therefore $I_d \subset \operatorname{Ann}_d(\Lambda^*)$.

Conversely, let $p \in \mathrm{Ann}_d(\Lambda^*)$. For all $\xi \in \mathcal{V}(I)$ we have $\mathbf{e}_\xi^{[2d]} \in \mathcal{L}_{2d}(\pm\mathbf{h})$. Since Λ^* is generic, we have $p \in \mathrm{Ann}_d(\mathbf{e}_\xi)$. In particular $\left\langle \mathbf{e}_\xi \middle| p \right\rangle = p(\xi) = 0$, and therefore p vanishes on all the points of $\mathcal{V}(I)$. Since I is real radical, $p \in I \cap \mathbb{R}[\mathbf{x}]_d = I_d$ and thus $\mathrm{Ann}_d(\Lambda^*) \subset I_d$, which concludes the proof. □

We describe now a property, known as *flat truncation*, which allows to test effectively if truncated moment sequences are coming from sums of evaluations.

Definition 3.4.17 (Flat truncation). Let $d_{\mathbf{g}} := \lceil \frac{1}{2} \max_{i=1,\dots,s} \deg(g_i) \rceil$. The *flat truncation* property holds for $\Lambda \in \mathcal{L}_d(\mathbf{g})$ at degree t if $t \le \frac{d}{2} - d_{\mathbf{g}}$ and

$$\operatorname{rank} H_\Lambda^t = \operatorname{rank} H_\Lambda^{t+d_{\mathbf{g}}}. \tag{3.1}$$

This definition coincides with the definition of flat truncation used in [CF96], [Lau09] or [Nie13b]. We investigate more in detail rank conditions for the moment matrix of $\Lambda \in \mathcal{L}_d(\mathbf{g})$.

Lemma 3.4.18. *If $\Lambda \in \mathcal{L}_d(\mathbf{g})$ is such that $\operatorname{rank} H_\Lambda^t = \operatorname{rank} H_\Lambda^{t+s} = r$ with $t+1 \le t+s \le \frac{d}{2}$, then*

$$\Lambda^{[t+s+\frac{d}{2}]} = \omega_1 \mathbf{e}_{\xi_1}^{[t+s+\frac{d}{2}]} + \cdots + \omega_r \mathbf{e}_{\xi_r}^{[t+s+\frac{d}{2}]}$$

for some points $\xi_i \in \mathbb{R}^n$ and weights $\omega_i > 0$, $i = 1,\dots,r$. Denoting $\Xi = \{\xi_1,\dots,\xi_r\}$, we also have $\mathrm{Ann}_{t+1}(\Lambda) = \mathcal{I}(\Xi)_{t+1}$ and $\mathcal{V}(\mathrm{Ann}_{t+1}(\Lambda)) = \Xi$ (or, in other words, $(\mathrm{Ann}_{t+1}(\Lambda)) = \mathcal{I}(\Xi)$).

Moreover, if $t \le \frac{d}{2} + s - \deg(\mathbf{g})$, where $\deg(\mathbf{g}) = \max_{i=1,\dots,s} \deg(g_i)$, the inclusion $\Xi \subset \mathcal{S}(\mathbf{g})$ holds true.

Proof. From [Lau09, th. 5.29], there exists unique $\Xi = \{\xi_1,\dots,\xi_r\} \subset \mathbb{R}^n$ and $\omega_1,\dots,\omega_r > 0$ such that $\Lambda^{[2(t+s)]} = \omega_1 \mathbf{e}_{\xi_1}^{[2(t+s)]} + \cdots + \omega_r \mathbf{e}_{\xi_r}^{[2(t+s)]}$, $(\mathrm{Ann}_{t+s}(\Lambda)) = \mathcal{I}(\Xi)$ and $\mathcal{V}(\mathrm{Ann}_{t+s}(\Lambda)) = \Xi$. In particular $(\mathrm{Ann}_{t+s}(\Lambda))$ is a zero dimensional ideal and $\mathrm{Ann}_{t+s}(\Lambda) \subset I(\Xi)_{t+s}$. Conversely, for any $h \in I(\Xi)_{t+s}$, we have

$$\left\langle \Lambda \middle| h^2 \right\rangle = \left\langle \Lambda^{[2(t+s)]} \middle| h^2 \right\rangle = \sum_{i=1}^r \omega_i \left\langle \mathbf{e}_{\xi_i}^{[2(t+s)]} \middle| h^2 \right\rangle = \sum_{i=1}^r \omega_i h^2(\xi_i) = 0.$$

Thus $h \in \mathrm{Ann}_{t+s}(\Lambda)$ (see see [Las+13, lem. 3.12]) and $I(\Xi)_{t+s} = \mathrm{Ann}_{t+s}(\Lambda)$.

As $\operatorname{rank} H_\Lambda^t = \operatorname{rank} H_\Lambda^{t+1} = r$, we deduce from above, that $(\mathrm{Ann}_{t+1}(\Lambda)) = I(\Xi)$ is generated in degree $\le t+1$ and that $\rho(\Xi) \le t+1$. Therefore Ξ has interpolator polynomials $u_1,\dots,u_r$ of degree $\le t$.

Let us show that the description of Λ on polynomials of degree $\le 2(t+s)$, can be extended to higher degree. For any $h \in \mathrm{Ann}_{t+s}(\Lambda) = I(\Xi)_{s+t}$, i.e. such that $\left\langle \Lambda \middle| h^2 \right\rangle = 0$, and any $p \in \mathbb{R}[\mathbf{x}]_{\frac{d}{2}}$ we have $\langle \Lambda | hp \rangle = 0$. This shows that $\Lambda \in (\mathcal{I}(\Xi)_{t+s+\frac{d}{2}})^\perp$. We deduce from Proposition 3.4.15 that $\Lambda^{[t+s+\frac{d}{2}]} \in \operatorname{cone}(\mathbf{e}_{\xi_1},\dots,\mathbf{e}_{\xi_r})^{[t+s+\frac{d}{2}]}$. This implies that $\Lambda^{[t+s+\frac{d}{2}]} = \omega_1 \mathbf{e}_{\xi_1}^{[t+s+\frac{d}{2}]} + \cdots + \omega_r \mathbf{e}_{\xi_r}^{[t+s+\frac{d}{2}]}$, evaluating $\langle \Lambda | u_i \rangle = \left\langle \Lambda^{[t+s+\frac{d}{2}]} \middle| u_i \right\rangle = \omega_i$ at the interpolator polynomials $u_1,\dots u_r$ of Ξ of degree $\le t$.

We show now that $\Xi = \{\xi_1,\dots,\xi_r\} \subset S$ if $t \le \frac{d}{2} + s - \deg(\mathbf{g})$. For $i = 1,\dots,r$ and $j = 1,\dots,m$ the polynomial $u_i^2 g_j$ has degree $\le 2t + \deg(\mathbf{g}) \le t + s + \frac{d}{2}$. Then we obtain:

$$0 \le \left\langle \Lambda \middle| u_i^2 g_j \right\rangle = \left\langle \Lambda^{[t+s+\frac{d}{2}]} \middle| u_i^2 g_j \right\rangle = \left\langle \omega_1 \mathbf{e}_{\xi_1}^{[t+s+\frac{d}{2}]} + \cdots + \omega_r \mathbf{e}_{\xi_r}^{[t+s+\frac{d}{2}]} \middle| u_i^2 g_j \right\rangle = g_j(\xi_i),$$

showing that $g_j(\xi_i) \ge 0$ for all i and j, i.e. $\Xi \subset \mathcal{S}(\mathbf{g})$. □

Remark. Lemma 3.4.18 can be used to test flat truncation in a simpler way when d is big, as we explain in the following. Assume for simplicity that $2d_{\mathbf{g}} = \deg(\mathbf{g})$. Then, if $\operatorname{rank} H_\Lambda^t = \operatorname{rank} H_\Lambda^{t+s}$ with $t \leq \frac{d}{2} + s - \deg(\mathbf{g})$, then $2(t+d_{\mathbf{g}}) = 2t + \deg(\mathbf{g}) \leq t + s + \frac{d}{2}$. Therefore, from Lemma 3.4.18 we deduce that Λ restricted to polynomials of degree $\leq 2(t+d_{\mathbf{g}})$ is equal to a sum of evaluations at points of S with positive weights, and the flat truncation is satisfied: $\operatorname{rank} H_\Lambda^t = \operatorname{rank} H_\Lambda^{t+d_{\mathbf{g}}}$. In particular, when $s=1$ and $d \geq t - 2 + 2\deg(\mathbf{g})$, $\operatorname{rank} H_\Lambda^t = \operatorname{rank} H_\Lambda^{t+1}$ implies $\operatorname{rank} H_\Lambda^t = \operatorname{rank} H_\Lambda^{t+d_{\mathbf{g}}}$.

We now show that we can use flat truncation to describe semialgebraic sets with a finite number of points.

Theorem 3.4.19. *If a positive linear functional $\Lambda^* \in \mathcal{L}_d(\mathbf{g})$ is such that $(\Lambda^*)^{[2(t+d_{\mathbf{g}})]}$ is generic in $\mathcal{L}_d(\mathbf{g})^{[2(t+d_{\mathbf{g}})]}$ (that is $\operatorname{Ann}_{t+d_{\mathbf{g}}}(\Lambda^*) \subset \operatorname{Ann}_{t+d_{\mathbf{g}}}(\Lambda)$ for all $\Lambda \in \mathcal{L}_d(\mathbf{g})$) and Λ^* satisfies the flat truncation property at degree $t \leq \frac{d}{2} - d_{\mathbf{g}}$, then:*

- *(i)* $S = \mathcal{S}(\mathbf{g}) = \{\xi_1, \ldots, \xi_r\}$ *is non-empty and finite;*
- *(ii)* $\mathcal{L}_d(\mathbf{g})^{[t+d_{\mathbf{g}}+\frac{d}{2}]} = \operatorname{cone}(\mathbf{e}_{\xi_1}, \ldots, \mathbf{e}_{\xi_r})^{[t+d_{\mathbf{g}}+\frac{d}{2}]}$;
- *(iii)* $t+1 \geq \rho(\xi_1, \ldots, \xi_r)$ *and* $\operatorname{Ann}_{t+1}(\Lambda^*) = \mathcal{I}(\xi_1, \ldots, \xi_r)_{t+1} = \mathcal{I}(S)_{t+1}$ *is the vanishing ideal of S truncated in degree $t+1$.*
- *(iv)* $\mathcal{I}(S)_{2(t+d_{\mathbf{g}})} \subset \overline{\mathcal{Q}_d(\mathbf{g})}$ *and* $(\operatorname{Ann}_{t+1}(\Lambda^*)) = \sqrt[\mathbb{R}]{\operatorname{supp} \mathcal{Q}(\mathbf{g})} = \mathcal{I}(S)$.

Proof. Let $\Lambda^* \in \mathcal{L}_d(\mathbf{g})$ be such that $(\Lambda^*)^{[2(t+d_{\mathbf{g}})]}$ is generic in $\mathcal{L}_d(\mathbf{g})^{[2(t+d_{\mathbf{g}})]}$, and assume that $\operatorname{rank} H_{\Lambda^*}^t = \operatorname{rank} H_{\Lambda^*}^{t+d_{\mathbf{g}}}$ with $t \leq \frac{d}{2} - d_{\mathbf{g}}$. By Lemma 3.4.18 applied with $s = d_{\mathbf{g}}$,

$$(\Lambda^*)^{[t+d_{\mathbf{g}}+\frac{d}{2}]} = \omega_1 \mathbf{e}_{\xi_1}^{[t+d_{\mathbf{g}}+\frac{d}{2}]} + \cdots + \omega_r \mathbf{e}_{\xi_r}^{[t+d_{\mathbf{g}}+\frac{d}{2}]}$$

with $\omega_i > 0$, $\Xi = \{\xi_1, \ldots, \xi_r\} \subset S$, $\operatorname{Ann}_{t+1}(\Lambda^*) = I(\Xi)_{t+1}$ and $(\operatorname{Ann}_{t+1}(\Lambda^*)) = I(\Xi)$.

Let $\mathbf{h} = h_1, \ldots, h_m \subset \operatorname{Ann}_{t+1}(\Lambda^*)$ be a graded basis of $I(\Xi)$ of degree $\leq t+1$. As $(\Lambda^*)^{[2(t+d_{\mathbf{g}})]}$ is generic, for any $\Lambda \in \mathcal{L}_d(\mathbf{g})$ we have $\operatorname{Ann}_{t+d_{\mathbf{g}}}(\Lambda^*) \subset \operatorname{Ann}_{t+d_{\mathbf{g}}}(\Lambda)$ and $\langle \Lambda | h_i^2 \rangle = 0$. Then for any $p \in \mathbb{R}[\mathbf{x}]_{d_{\mathbf{g}}+\frac{d}{2}}$ we have $\langle \Lambda | h_i p \rangle = 0$, proving that $\Lambda \in (\mathbf{h})^\perp_{t+d_{\mathbf{g}}+\frac{d}{2}} = (\mathcal{I}(\Xi)_{t+d_{\mathbf{g}}+\frac{d}{2}})^\perp$, i.e. $\mathcal{L}_d(\mathbf{g})^{[t+d_{\mathbf{g}}+\frac{d}{2}]} \subset (\mathcal{I}(\Xi)_{t+d_{\mathbf{g}}+\frac{d}{2}})^\perp$. We deduce from Proposition 3.4.15 that $\Lambda^{[t+d_{\mathbf{g}}+\frac{d}{2}]} \in \operatorname{cone}(\mathbf{e}_{\xi_1}, \ldots, \mathbf{e}_{\xi_r})^{[t+d_{\mathbf{g}}+\frac{d}{2}]}$. This shows that $\mathcal{L}_d(\mathbf{g})^{[t+d_{\mathbf{g}}+\frac{d}{2}]} \subset \operatorname{cone}(\mathbf{e}_{\xi_1}, \ldots, \mathbf{e}_{\xi_r})^{[t+d_{\mathbf{g}}+\frac{d}{2}]}$. On the other hand the inclusion $\mathcal{L}_d(\mathbf{g})^{[t+d_{\mathbf{g}}+\frac{d}{2}]} \supset \operatorname{cone}(\mathbf{e}_{\xi_1}, \ldots, \mathbf{e}_{\xi_r})^{[t+d_{\mathbf{g}}+\frac{d}{2}]}$ holds true since $\Xi \subset S$. Therefore

$$\mathcal{L}_d(\mathbf{g})^{[t+d_{\mathbf{g}}+\frac{d}{2}]} = \operatorname{cone}(\mathbf{e}_{\xi_1}, \ldots, \mathbf{e}_{\xi_r})^{[t+d_{\mathbf{g}}+\frac{d}{2}]}.$$

Let us show that $\Xi = S$. For $\zeta \in S$ we have $\mathbf{e}_\zeta^{[t+d_{\mathbf{g}}+\frac{d}{2}]} \in \mathcal{L}_d(\mathbf{g})^{[t+d_{\mathbf{g}}+\frac{d}{2}]} \subset (\mathbf{h})^\perp_{t+d_{\mathbf{g}}+\frac{d}{2}}$, and thus for $i = 1, \ldots, m$, $\langle \mathbf{e}_\zeta | h_i \rangle = h_i(\zeta) = 0$. This shows that ζ is a root of $\mathbf{h}$ and thus $\zeta \in \Xi$. We conclude that $\Xi = \{\xi_1, \ldots, \xi_r\} = S$.

The inclusion $\mathcal{I}(S)_{2(t+d_{\mathbf{g}})} \subset \overline{\mathcal{Q}_d(\mathbf{g})}$ follows from $\mathcal{L}_d(\mathbf{g})^{[t+d_{\mathbf{g}}+\frac{d}{2}]} \subset (\mathbf{h})^\perp_{t+d_{\mathbf{g}}+\frac{d}{2}}$. Indeed $2(t+d_{\mathbf{g}}) \leq t + d_{\mathbf{g}} + \frac{d}{2}$ and thus $\mathcal{L}_d(\mathbf{g})^{[2(t+d_{\mathbf{g}})]} \subset (\mathbf{h})^\perp_{2(t+d_{\mathbf{g}})}$. Now notice that $(\mathcal{L}_d(\mathbf{g})^{[2(t+d_{\mathbf{g}})]})^\vee \subset \overline{\mathcal{Q}_d(\mathbf{g})}$, using convex duality. Therefore dualizing $\mathcal{L}_d(\mathbf{g})^{[2(t+d_\mathbf{g})]} \subset (\mathbf{h})_{2t}^{\perp}$ we obtain the desired inclusion. Moreover $\mathcal{I}(S)_{2(t+d_\mathbf{g})} \subset \overline{\mathcal{Q}_d(\mathbf{g})} \cap -\overline{\mathcal{Q}_d(\mathbf{g})} \subset \operatorname{supp} \widetilde{Q} = \sqrt[\mathbb{R}]{\operatorname{supp} Q}$, by Theorem 3.4.3, and finally:

$$(\operatorname{Ann}_{t+1}(\Lambda^*)) = \mathcal{I}(S) = (\mathcal{I}(S)_{2(t+d_\mathbf{g})}) \subset \sqrt[\mathbb{R}]{\operatorname{supp} \mathcal{Q}(\mathbf{g})} \subset \sqrt[\mathbb{R}]{\operatorname{supp} \mathcal{O}(\mathbf{g})} = \mathcal{I}(S),$$

where the last equality is the Real Nullstellenstatz, Theorem 1.1.25. This shows that $(\operatorname{Ann}_{t+1}(\Lambda^*)) = \sqrt[\mathbb{R}]{\operatorname{supp} \mathcal{Q}(\mathbf{g})} = \mathcal{I}(S)$. □

This theorem tells us that if the flat truncation property holds at degree $t \leq \frac{d}{2} - d_\mathbf{g}$, then any element of $\mathcal{L}_d(\mathbf{g})$ truncated in degree $t + \frac{d}{2} + d_\mathbf{g}$ coincides with a positive measure supported on $S = \{\xi_1, \ldots, \xi_r\}$. Moreover, when S is finite point (iv) of Theorem 3.4.19 implies that flat truncaiton can be seen as a test to verify if $\mathcal{I}(S) = \sqrt[\mathbb{R}]{\operatorname{supp} \mathcal{Q}(\mathbf{g})}$ or not (recall that, from the Real Nullstelensatz, in general we need the preordering to describe the vanishing ideal of a semialgebraic set: $\mathcal{I}(S) \in \sqrt[\mathbb{R}]{\operatorname{supp} \mathcal{O}(\mathbf{g})}$).

In the following theorem we investigate the converse: we show that when $\operatorname{supp}(Q)$ is a zero-dimensional ideal (and thus S is finite), the rank condition is satisfied for any moment matrix.

Theorem 3.4.20. *Suppose that* $\dim \frac{\mathbb{R}[\mathbf{x}]}{\operatorname{supp} \mathcal{Q}(\mathbf{g})} = 0$. *Then* $S = \mathcal{S}(\mathbf{g})$ *is finite and there exists* $d \geq 2(\rho - 1 + d_\mathbf{g})$ *such that* $\mathcal{I}(S)_{2(\rho-1+d_\mathbf{g})} \subset \operatorname{supp} \overline{\mathcal{Q}_d(\mathbf{g})}$, *where* $\rho = \rho(S)$ *is the regularity of* S, *and for any* $\Lambda \in \mathcal{L}_d(\mathbf{g})$ *the flat truncation property holds at degree* $\rho - 1$.

Proof. Let $I = \operatorname{supp} \mathcal{Q}(\mathbf{g})$ and $J = \sqrt[\mathbb{R}]{\operatorname{supp} \mathcal{Q}(\mathbf{g})}$. From Lemma 1.1.24 we deduce $\dim \frac{\mathbb{R}[\mathbf{x}]}{J} = \dim \frac{\mathbb{R}[\mathbf{x}]}{I} = 0$ and by Theorem 1.1.26 we have $\mathcal{I}(\mathcal{S}(\mathbf{g})) = \sqrt[\mathbb{R}]{\operatorname{supp} \mathcal{Q}(\mathbf{g})} = J$. Then $\mathcal{V}_\mathbb{R}(J) = \mathcal{V}_\mathbb{R}(\mathcal{I}(\mathcal{S}(\mathbf{g}))) = \mathcal{S}(\mathbf{g}) = \{\xi_1, \ldots, \xi_r\}$ is finite.

We choose a graded basis $\mathbf{h}$ of J with $\deg \mathbf{h} \leq \rho = \rho(\xi_1, \ldots, \xi_r)$ (by Proposition 3.4.14). By Corollary 3.4.7, there exists $d \in \mathbb{N}$ such that $\mathcal{I}(S)_{2(\rho-1+d_\mathbf{g})} \subset \operatorname{supp} \overline{\mathcal{Q}_d(\mathbf{g})}$. From Corollary 3.4.7 and Proposition 3.4.15 we deduce that positive linear functionals in $\mathcal{L}_d(\mathbf{g})$ restricted to degree $\leq 2(\rho - 1 + d_\mathbf{g})$ are conic combinations of evaluations at $\xi_1, \ldots, \xi_r$:

$$\mathcal{L}_d(\mathbf{g})^{[2(\rho-1+d_\mathbf{g})]} \subset \mathcal{L}_{2(\rho-1+d_\mathbf{g})}(\pm\mathbf{h}) = \mathcal{L}_{2(\rho-1+d_\mathbf{g})}(J_{2(\rho-1+d_\mathbf{g})}) = \operatorname{cone}(\mathbf{e}_{\xi_1}, \ldots, \mathbf{e}_{\xi_r})^{[2(\rho-1+d_\mathbf{g})]},$$

and for all $\Lambda \in \mathcal{L}_d(\mathbf{g})$, we have $\operatorname{rank} H_\Lambda^{\rho-1} = \operatorname{rank} H_\Lambda^{\rho-1+d_\mathbf{g}}$. □

Theorem 3.4.20 says that if $\dim \frac{\mathbb{R}[\mathbf{x}]}{\operatorname{supp} \mathcal{Q}(\mathbf{g})} = 0$ then the minimal order for which we have flat truncation is not bigger than $d \geq 2(\rho - 1 + d_\mathbf{g})$ such that $\mathcal{I}(S)_{2(\rho-1+d_\mathbf{g})} \subset \operatorname{supp} \overline{\mathcal{Q}_d(\mathbf{g})}$. This degree is related to the minimal d for which $\mathcal{I}(S) = \sqrt[\mathbb{R}]{\operatorname{supp} \mathcal{Q}(\mathbf{g})}$ is generated by $\operatorname{supp} \overline{\mathcal{Q}_d(\mathbf{g})}$, that is, the minimal degree d such that $I(S)_{\rho-1+d_\mathbf{g}} \subset \operatorname{Ann}_{\frac{d}{2}}(\Lambda^*)$ for a generic $\Lambda^* \in \mathcal{L}_d(\mathbf{g})$. Moreover, as in the remark after Lemma 3.4.18, we can replace $\rho - 1 + d_\mathbf{g}$ with ρ if d is big enough.

Theorem 3.4.20 and Theorem 3.4.19 show that if $\dim \frac{\mathbb{R}[\mathbf{x}]}{\operatorname{supp} \mathcal{Q}(\mathbf{g})} = 0$ then S is a finite set of points and for a high enough degree d, all moment sequences in $\mathcal{L}_d(\mathbf{g})$, truncated in degree twice the regularity are coming from a weighted sum of Dirac measures at these points. In particular, it is possible to recover all the points in S from a generic truncated moment sequence, see [HL05], [ABM15] and [Mou18].

Remark. There exist examples with $\mathcal{S}(\mathbf{g})$ finite and $\dim \frac{\mathbb{R}[\mathbf{x}]}{\operatorname{supp} \mathcal{Q}(\mathbf{g})} > 1$, see Example 3.4.4. However, the hypotheses

(i) $\dim \frac{\mathbb{R}[\mathbf{x}]}{\operatorname{supp} \mathcal{Q}(\mathbf{g})} = 0$; and

(ii) $\mathcal{S}(\mathbf{g})$ is finite and $\dim \frac{\mathbb{R}[\mathbf{x}]}{\operatorname{supp} \mathcal{Q}(\mathbf{g})} \leq 1$;

are equivalent: (i) ⇒ (ii) is shown in the proof of Theorem 3.4.20, while (ii) ⇒ (i) follows from $\mathcal{I}(\mathcal{S}(\mathbf{g})) = \sqrt[\mathbb{R}]{\operatorname{supp} \mathcal{Q}(\mathbf{g})}$ (see Theorem 1.1.26).

Results related to Theorem 3.4.20 and Theorem 3.4.19 were obtained in [LLR08] and [Las+13], where they focus on the case of equations $\mathbf{h}$ defining a finite real variety. They prove that, for d big enough and for every positive linear functional $\Lambda \in \mathcal{L}_{2d}(\pm\mathbf{h})$, the flat truncation property holds for H_Λ^d, and that $\Lambda^{[2d]}$ is a conic linear combination of evaluations at the points of $\mathcal{V}_\mathbb{R}(\mathbf{h})$. This can be deduced from Theorem 3.4.20, since in the case where $\mathcal{V}_\mathbb{R}(\mathbf{h}) = \{\xi_1, \ldots, \xi_r\}$ is non-empty and finite, $\dim \frac{\mathbb{R}[\mathbf{x}]}{\operatorname{supp} \mathcal{Q}(\pm\mathbf{h})} = 0$.

In [LLR08, rem. 4.9] it is also mentioned that the same can be proved for a preordering defining a finite semialgebraic set. This result can also be deduced from Theorem 3.4.20, since when $S = S(\mathbf{g}) = \{\xi_1, \ldots, \xi_r\}$ is non-empty and finite, we have by the Real Nullstellensatz: $\dim \frac{\mathbb{R}[\mathbf{x}]}{\operatorname{supp} \mathcal{O}(\mathbf{g})} = \dim \frac{\mathbb{R}[\mathbf{x}]}{\sqrt[\mathbb{R}]{\operatorname{supp} \mathcal{O}(\mathbf{g})}} = \dim \frac{\mathbb{R}[\mathbf{x}]}{\mathcal{I}(\mathcal{S}(\mathbf{g}))} = 0$.

But Theorem 3.4.20 is more general, as shown by the following example of a quadratic module, whose support is zero dimensional, but that is not a preordering.

Example 3.4.21 ([Mar08, ex. 7.4.5 (1)])**.** Let $Q = \mathcal{Q}(x, y, 1-x, 1-y, -x^4, -y^4) \subset \mathbb{R}[x, y]$. In this case $\operatorname{supp} Q$, which contains x^4 and y^4, is zero dimensional and Q is not a preordering since $xy \notin Q$ (see [Mar08, ex. 7.4.5 (1)]). Therefore, Theorem 3.4.20 applies in this case, but the results cannot be deduced from [LLR08] or [Las+13].

As we will see, in Polynomial Optimization problems, flat truncation implies moment exactness and thus finite convergence. Moreover, it allows extracting the minimizers from an optimal sequence.

3.5 Flat truncation in polynomial optimization problems

In this section, we analyze when flat truncation occurs in the Polynomial Optimization Problem, which consists of minimizing $f \in \mathbb{R}[\mathbf{x}]$ on the basic semialgebraic set $S = \mathcal{S}(\mathbf{g})$ where $\mathbf{g} = g_1, \ldots, g_s$ is a tuple of polynomials. Recall that we denote f^* the minimum of f on S. We will consider the semialgebraic set $S^{\min} = \mathcal{S}(\mathbf{g}, \pm(f - f^*)) = \mathcal{S}(\mathbf{g}) \cap \{x \in \mathbb{R}^n \mid f(x) = f^*\}$ and assume that it is nonempty.

3.5.1 Flat truncation degree

Hereafter, we analyze the degree at which flat truncation holds and yields the minimizers.

Theorem 3.5.1. *Consider the problem of minimizing f on $\mathcal{S}(\mathbf{g})$. If the flat truncation property holds for a generic $\Lambda^* \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ at a degree t such that $\deg(f) - d_\mathbf{g} - d \leq t \leq d - d_\mathbf{g}$, then:*

(i) $f^* = f^*_{\mathrm{Mom},d}$ (i.e. we have moment finite convergence);

(ii) the set of minimizers $S^{\min} = \{\xi_1, \ldots, \xi_r\}$ is non-empty and finite;

(iii) $\ker H_{\Lambda^*}^{t+1} = \mathrm{Ann}_{t+1}(\Lambda^*) = \mathcal{I}(S^{\min})_{t+1}$ *(i.e. the kernel of the truncated moment matrix equals the truncated ideal of the minimizers) and* $\mathcal{V}(\mathrm{Ann}_{t+1}(\Lambda^*)) = S^{\min}$*;*

(iv) $\mathcal{L}_d^{\min}(\mathbf{g})^{[t+d_{\mathbf{g}}+d]} = \mathrm{cone}(\mathbf{e}_{\xi_1},\dots,\mathbf{e}_{\xi_r})^{[t+d_{\mathbf{g}}+d]}$ *(i.e. all the minimizing truncated feasible moment sequences are conic sums of evaluations at the minimizers);*

(v) the moment hierarchy is exact.

Proof. Let $\Lambda^* \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ be generic such that $\operatorname{rank} H_{\Lambda^*}^t = \operatorname{rank} H_{\Lambda^*}^{t+d_{\mathbf{g}}}$ with $\deg(f) \le t+d_{\mathbf{g}}+d$ and $t+d_{\mathbf{g}} \le d$. Then by Lemma 3.4.18, $(\Lambda^*)^{[t+d_{\mathbf{g}}+d]} = \sum_{i=1}^r \omega_i \mathbf{e}_{\xi_i}^{[t+d_{\mathbf{g}}+d]}$ with $\xi_i \in S = \mathcal{S}(\mathbf{g})$, $\omega_i > 0$, $\mathrm{Ann}_{t+1}(\Lambda^*) = \mathcal{I}(\xi_1,\dots\xi_r)_{t+1} = \mathcal{I}(\Xi)_{t+1}$ and $\mathcal{V}(\mathrm{Ann}_{t+1}(\Lambda^*)) = \Xi$. Notice that $f(\xi_i) \ge f^*$ since $\xi_i \in S$.

We show now that $S^{\min} = \Xi$. As $\langle \Lambda^* | 1\rangle = 1$ we have $\sum_{i=1}^r \omega_i = 1$. Moreover $f^*_{\mathrm{Mom},d} = \langle \Lambda^* | f\rangle \le f^*$ and since $\deg(f) \le t+d_{\mathbf{g}}+d$ we obtain:

$$f^* \ge \langle \Lambda^* | f\rangle = \left\langle (\Lambda^*)^{[t+d_{\mathbf{g}}+d]} \middle| f\right\rangle = \sum_{i=1}^r \omega_i \left\langle \mathbf{e}_{\xi_i}^{[t+d_{\mathbf{g}}+d]} \middle| f\right\rangle = \sum_{i=1}^r \omega_i f(\xi_i) \ge f^*.$$

This implies that $f(\xi_i) = f^*$ for $i = 1,\dots,r$. Therefore $f^* = f^*_{\mathrm{Mom},d}$ and $S^{\min} \supset \Xi$.

From Proposition 3.4.10 we have that $\Lambda^* \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ generic implies that $(\Lambda^*)^{[2(t+d_{\mathbf{g}})]}$ is generic in $\mathcal{L}_{2d}^{\min}(\mathbf{g})^{[2(t+d_{\mathbf{g}})]}$. Moreover $(\Lambda^*)^{[2(t+d_{\mathbf{g}})]} = \sum_{i=1}^r \omega_i \mathbf{e}_{\xi_i}^{[2(t+d_{\mathbf{g}})]} \in \mathcal{L}_{2d}^{(1)}(\mathbf{g}, \pm(f-f^*))^{[2(t+d_{\mathbf{g}})]}$ since $\Xi \subset S^{\min} = \mathcal{S}(\mathbf{g}, \pm(f-f^*))$. Then, as $\mathcal{L}_{2d}^{(1)}(\mathbf{g}, \pm(f-f^*)) \subset \mathcal{L}_{2d}^{\min}(\mathbf{g})$ and $(\Lambda^*)^{[2(t+d_{\mathbf{g}})]}$ is generic in $\mathcal{L}_{2d}^{\min}(\mathbf{g})^{[2(t+d_{\mathbf{g}})]}$, we have

$$\forall \Lambda \in \mathcal{L}_{2d}(\mathbf{g}, \pm(f-f^*)) \quad \mathrm{Ann}_{t+d_{\mathbf{g}}}(\Lambda^*) \subset \mathrm{Ann}_{t+d_{\mathbf{g}}}(\Lambda),$$

i.e. $(\Lambda^*)^{[2(t+d_{\mathbf{g}})]}$ is generic in $\mathcal{L}_{2d}(\mathbf{g}, \pm(f-f^*))^{[2(t+d_{\mathbf{g}})]}$. We can then conclude from Theorem 3.4.19 that $S^{\min} = \Xi$ and $\mathcal{L}_d^{\min}(\mathbf{g})^{[t+d_{\mathbf{g}}+d]} = \mathrm{cone}(\mathbf{e}_{\xi_1},\dots,\mathbf{e}_{\xi_r})^{[t+d_{\mathbf{g}}+d]}$.

Finally we show moment exactness. For every $d' \ge d$ and $\Lambda \in \mathcal{L}_{2d'}^{\min}(\mathbf{g})$, we have $\Lambda^{[2d]} \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ since $\langle \Lambda | f\rangle = f^*$. Therefore Λ has flat truncation in degree t and by Lemma 3.4.18, $\Lambda^{[t+d_{\mathbf{g}}+d']}$ is coming from a convex sum of Dirac measures at points in S (that are the minimizers $\xi_1,\dots,\xi_r$). This shows that the moment relaxation is exact, since increasing d' we increase also the truncation degree where Λ coincides with a weighted sum of evaluations at the minimizers. □

Theorem 3.5.1 slightly relaxes previous degree conditions. In [Lau09, th. 6.18], the degree condition is $\deg(f) \le 2t + 2d_{\mathbf{g}} \le t + d_{\mathbf{g}} + d$. It also shows that the kernel of the moment matrix of a generic truncated moment sequence, $\mathrm{Ann}_{t+1}(\Lambda^*)$, is the truncated vanishing ideal of the minimizers and that the relaxation is exact. This means that any element in $\mathcal{L}_{2d}^{\min}(\mathbf{g})$ truncated in any degree t is coming from a measure, provided $d \ge t$ is big enough.

A key ingredient in this analysis is Lemma 3.4.18. From Lemma 3.4.18 and the remark after it, the results of Theorem 3.5.1 hold true, if we replace the condition $\operatorname{rank} H_{\Lambda^*}^t = \operatorname{rank} H_{\Lambda^*}^{t+d_{\mathbf{g}}}$ with $\operatorname{rank} H_{\Lambda^*}^t = \operatorname{rank} H_{\Lambda^*}^{t+1}$ and d big enough.

We show in Example 3.5.2 that the condition $\operatorname{rank} H_{\Lambda^*}^t = \operatorname{rank} H_{\Lambda^*}^{t+1}$ is in general not sufficient to conclude that the points extracted from the moment matrix are inside the semialgebraic set. To the best of our knowledge, this is the first example where such a pathological behaviour is explicit.

Example 3.5.2. We consider the problem of minimizing $f = (1+x)(x-1)^2$ on $\mathcal{S}(1-x^2, -x^3) = [-1, 0]$. Notice that the SoS hierarchy is exact, since $f^* = 0$ and:

$$(1+x)(x-1)^2 = \frac{1}{2}\Big((1+x)^2 + 1 - x^2\Big)(x-1)^2 \in \mathcal{Q}_4(1-x^2, -x^3).$$

This implies that $f^*_{\mathrm{SoS},2} = f^*_{\mathrm{Mom},2} = f^*$. The only minimizer of f on S is -1, and $\mathcal{I}(-1) = (x+1)$: therefore we would expect to get flat truncation at degree zero for a generic element, and in particular $\operatorname{rank} H^0_{\Lambda^*} = \operatorname{rank} H^1_{\Lambda^*} = 1$. But this is not the case if we consider the moment relaxation of order 2. Indeed, an explicit calculation shows that $\Lambda = \frac{1}{2}(\mathbf{e}^{[4]}_{-1} + \mathbf{e}^{[4]}_{1}) \in \mathcal{L}^{\min}_4(\mathbf{g})$, and $\operatorname{rank} H^1_{\Lambda} = \operatorname{rank} H^2_{\Lambda} = 2$. Therefore, a generic $\Lambda^* \in \mathcal{L}^{\min}_4(\mathbf{g})$ cannot satisfy the rank condition for $t = 0$. More precisely, it is possible to show that $\mathcal{L}^{\min}_4(\mathbf{g}) = \operatorname{conv}\Big(\mathbf{e}^{[4]}_{-1}, \frac{1}{2}(\mathbf{e}^{[4]}_{-1} + \mathbf{e}^{[4]}_{1})\Big)$ (it is important to exploit the equation $1 - \Lambda_1 - \Lambda_0 + \Lambda_3 = 0$ in the definition of $\mathcal{L}^{\min}_4(\mathbf{g})$, arising from $\langle \Lambda | f \rangle = f^*_{\mathrm{Mom},d} = 0$). Therefore a generic $\Lambda^* \in \mathcal{L}^{\min}_4(\mathbf{g})$ will also satisfy $\operatorname{rank} H^1_{\Lambda} = \operatorname{rank} H^2_{\Lambda} = 2$. Notice that this is an example where an extremal point is generic (see Definition 3.4.9): $\frac{1}{2}(\mathbf{e}^{[4]}_{-1} + \mathbf{e}^{[4]}_{1})$ is extremal in $\mathcal{L}^{\min}_4(\mathbf{g})$, but it is also generic since $\operatorname{rank} H^2_{\frac{1}{2}(\mathbf{e}^{[4]}_{-1}+\mathbf{e}^{[4]}_{1})} = 2$.

We confirm numerically the computation above, using the package `MomentTools.jl` to compute f^* and a generic $\Lambda^* \in \mathcal{L}^{\min}_4(\mathbf{g})$: the moments that we obtain are

$$\Lambda^*_0 = 0.999999998978497, \quad \Lambda^*_1 = -0.353032474967529 \quad \Lambda^*_2 = 0.999847411529907$$
$$\Lambda^*_3 = -0.353185157145022 \quad \Lambda^*_4 = 0.999694736472143.$$

We compute the singular values of $H^0_{\Lambda^*}$, $H^1_{\Lambda^*}$ and $H^2_{\Lambda^*}$ to have a numerically stable indication of the ranks:

Sing. Val. of $H^0_{\Lambda^*}$: 0.9999999989784975

Sing. Val. of $H^1_{\Lambda^*}$: 1.352956188465637, 0.6468912220427679

Sing. Val. of $H^2_{\Lambda^*}$: 2.2063794508570065, 0.7931627759613444, $7.983780245045715 \cdot 10^{-8}$

This confirms the theoretical description and shows that the rank condition is numerically satisfied for $t = 1$. The points extracted from the matrix are $\xi_1 \approx 0.9997640487211856$ and $\xi_2 \approx -1.0000000483192044$: notice that $\xi_1 \notin S$. This happens because the condition $\operatorname{rank} H^t_{\Lambda^*} = \operatorname{rank} H^{t+d_{\mathbf{g}}}_{\Lambda^*}$ is not satisfied (it is not possible to compute $H^{t+d_{\mathbf{g}}}_{\Lambda^*} = H^3_{\Lambda^*}$ since $\Lambda^* \in \mathcal{L}_{2d}(\mathbf{g}) = \mathcal{L}_4(\mathbf{g})$ and $3 = t + d_{\mathbf{g}} > d = 2$).

On the other hand, if we increase the order of the relaxation and compute $\Lambda^* \in \mathcal{L}^{\min}_6(\mathbf{g})$ generic, we can verify flat truncation for $t = 0$ and the only point extracted is -1. Moreover notice, from Lemma 3.4.18 applied with $s = 1$ and the remark below, that it is enough to check $\operatorname{rank} H^0_{\Lambda^*} = \operatorname{rank} H^1_{\Lambda^*}$ to verify that $\operatorname{rank} H^t_{\Lambda^*} = \operatorname{rank} H^{t+d_{\mathbf{g}}}_{\Lambda^*}$, since the condition $0 = t \le d + s - \deg(\mathbf{g}) = 1$ is satisfied.

We have seen that flat truncation implies moment exactness and a finite set of minimizers. We show now that, under the assumption of moment finite convergence, flat truncation is equivalent to a zero dimensional support for the quadratic module $Q + (f - f^*)$ defining the minimizers.

We first need a technical lemma, that will be important to investigate the relationship between $\mathcal{L}^{\min}_{2d}(\mathbf{g})$ and $\mathcal{L}^{(1)}_{2d}(\mathbf{g}, \pm(f - f^*))$. Indeed, notice that $\mathcal{L}^{(1)}_{2d}(\mathbf{g}, \pm(f - f^*)) \subset \mathcal{L}^{\min}_{2d}(\mathbf{g})$, by definition, but the converse inclusion is not true in general, since for $\Lambda \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ we only have $\langle \Lambda | f \rangle = f^*$, and not $f - f^* \in \mathrm{Ann}_{d-\frac{\deg(f)}{2}}(\Lambda)$ (or, in other words, there may exist h such that $\langle \Lambda | h(f - f^*) \rangle \neq 0$).

Lemma 3.5.3. *Let $f \in \mathcal{Q}_{2k}(\mathbf{g})$, $\Lambda \in \mathcal{L}_{2d}(\mathbf{g})$ and $t \in \mathbb{N}$ with $0 \le t \le d - k$. Then $\langle \Lambda | f \rangle = 0$ implies for all $q \in \mathbb{R}[x]_t$, $\langle \Lambda | qf \rangle = 0$. In other words, $f \in \mathrm{Ann}_t(\Lambda)$.*

Proof. We set $g_0 = 1$ for notation convenience. Let $f = \sum_i s_i g_i = \sum_{i,j} p_{i,j}^2 g_i \in \mathcal{Q}_{2k}(\mathbf{g})$, that is $\deg p_{i,j}^2 g_i \le 2k$. We want to prove that for all $q \in \mathbb{R}[\mathbf{x}]$ such that $\deg(q) \le t$ we have $\langle \Lambda | qf \rangle = 0$. In particular, it is enough to prove that:

$$\left\langle \Lambda \middle| q p_{i,j}^2 g_i \right\rangle = 0 \text{ for all } i, j \text{ and } q \in \mathbb{R}[\mathbf{x}]. \tag{3.2}$$

Now, notice that $\langle \Lambda | f \rangle = 0$ implies $\left\langle \Lambda \middle| p_{i,j}^2 g_i \right\rangle = 0$ for all i, j, and consider for all $T \in \mathbb{R}$ and $h \in \mathbb{R}[\mathbf{x}]_{t+\deg p_{i,j}}$:

$$0 \le \left\langle \Lambda \middle| (p_{i,j} - Th)^2 g_i \right\rangle = T^2 \left\langle \Lambda \middle| h^2 g_i \right\rangle + 2T \left\langle \Lambda \middle| h p_{i,j} g_i \right\rangle$$

(we can apply Λ to $(p_{i,j} - Th)^2 g_i$ since $\deg((p_{i,j} - Th)^2 g_i) \le 2t + 2k \le 2d$). The polynomial $T \mapsto T^2 \left\langle \Lambda \middle| h^2 g_i \right\rangle + 2T \left\langle \Lambda \middle| h p_{i,j} g_i \right\rangle$ has therefore a double root at $T = 0$, and this implies $\left\langle \Lambda \middle| h p_{i,j} g_i \right\rangle = 0$ for all $h \in \mathbb{R}[\mathbf{x}]_{t+\deg p_{i,j}}$. If we subsitute $h = q p_{i,j}$, we deduce eq. (3.2), and thus $f \in \mathrm{Ann}_t(\Lambda)$. □

For a concrete example where the difference between $\langle \Lambda | f \rangle = f^*$ and $f - f^* \in \mathrm{Ann}_{d-\frac{\deg(f)}{2}}(\Lambda)$ is explicit, consider Example 3.2.2. In this case, for every $\Lambda(M)$ defined there, we have $\langle \Lambda(M) | f - f^* \rangle = \left\langle \Lambda(M) \middle| x^3 \right\rangle = 0$, but $\left\langle \Lambda(M) \middle| x^{2d-3}(f - f^*) \right\rangle = \left\langle \Lambda(M) \middle| x^{2d} \right\rangle = M$. Therefore, for $M > 0$, $\Lambda(M) \in \mathcal{L}_2 d^{\min}(\mathbf{g}) \setminus \mathcal{L}_{2d}^{(1)}(\mathbf{g}, \pm(f - f^*))$.

We can now prove the equivalence between the flat truncation and the zero dimensional support for the quadratic module $Q + (f - f^*)$ defining the minimizers.

Theorem 3.5.4. *Assume that we have moment finite convergence. Then $\dim \frac{\mathbb{R}[\mathbf{x}]}{\mathrm{supp}(Q+(f-f^*))} = 0$ if and only if there exists d such that a generic $\Lambda^* \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ has flat truncation.*

In particular, if $\rho = \rho(S^{\min})$, $D = \max(d_{\mathbf{g}}, \lceil \frac{\deg(f)}{2} \rceil)$ and $\delta \in \mathbb{N}$ is such that $f - f^* \in \overline{\mathcal{Q}_{2\delta}(\mathbf{g})}$, flat truncation happens for $\Lambda^* \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ generic at degree $\rho - 1$ when d is such that:

(i) $(\sqrt[\mathbb{R}]{\mathrm{supp}\, Q(\mathbf{g})})_{2\delta+2\rho+2D-\deg(f)-2} \subset \overline{\mathcal{Q}_{2d}(\mathbf{g})}$;

(ii) $\mathcal{I}(S^{\min})_{2\rho+2D-2} \subset \overline{\mathcal{Q}_{2d}(\mathbf{g}) + (f - f^*)_{2d}}$;

(iii) $\delta + 2\rho + 2D - \deg(f) - 2 \le d$.

Proof. Let us assume without loss of generality that $f^* = 0$.

We first show that flat truncation implies $\dim \frac{\mathbb{R}[\mathbf{x}]}{\mathrm{supp}(Q+(f))} = 0$. As in the proof of Theorem 3.5.1, if $\Lambda^* \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ is generic satisfying flat truncation at degree t then $(\Lambda^*)^{[2(t+d_{\mathbf{g}})]}$ is a generic element of $\mathcal{L}_{2d}(\mathbf{g}, \pm f)^{[2(t+d_{\mathbf{g}})]}$. Since the flat truncation property is satisfied, we conclude from Theorem 3.4.19 that $\sqrt[\mathbb{R}]{\mathrm{supp}(Q + (f))} = (\mathrm{Ann}_{t+1}(\Lambda^*)) = \mathcal{I}(S^{\min})$ and finally, applying Lemma 1.1.24, $\dim \frac{\mathbb{R}[\mathbf{x}]}{\mathrm{supp}(Q+(f))} = \dim \frac{\mathbb{R}[\mathbf{x}]}{\mathcal{I}(S^{\min})} = 0$.

Conversely, if $\dim \frac{\mathbb{R}[\mathbf{x}]}{\operatorname{supp}(Q+(f))} = 0$, we deduce from Theorem 3.4.20 that the flat truncation property is satisfied for any $\Lambda \in \mathcal{L}_{2d}(\mathbf{g}, \pm f)$ at degree $\rho - 1 = \rho(S(\mathbf{g}, \pm(f - f^*)) - 1 = \rho(S^{\min}) - 1$ for d such that $\mathcal{I}(S^{\min})_{2(\rho-1+D)} \subset \overline{\mathcal{Q}_{2d}(\mathbf{g}) + (f)_{2d}}$. Let $a = 2\rho - 2 + 2D$ and $\Lambda^* \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ generic. We want to show that $(\Lambda^*)^{[a]} \in \mathcal{L}_{2d}(\mathbf{g}, \pm f)^{[a]}$, so that we can conclude using Theorem 3.4.20. Since $\Lambda^* \in \mathcal{L}_{2d}^{\min}(\mathbf{g}) \subset \mathcal{L}_{2d}(\mathbf{g})$, it is sufficient to prove that:

$$\langle \Lambda^* | qf \rangle = 0 \text{ for all } q \text{ of degree} \leq a - \deg(f). \tag{3.3}$$

We prove now (3.3), starting from $\langle \Lambda^* | f \rangle = f^* = 0$. Moment finite convergence implies that $\langle \Lambda | f \rangle \geq 0$ for all $\Lambda \in \mathcal{L}_{2d}(\mathbf{g})$, and therefore $f \in \mathcal{L}_{2d}(\mathbf{g})^{\vee} = \overline{\mathcal{Q}_{2d}(\mathbf{g})}$. Let $\delta \leq d$ be minimal such that $f \in \overline{\mathcal{Q}_{2\delta}(\mathbf{g})}$ and let $\mathbf{h} = h_1, \dots h_m$ be a graded basis of $\sqrt[\mathbb{R}]{\operatorname{supp} Q}$. From [Mar08, lemma 4.1.4] we deduce that $\mathcal{Q}_{2\delta}(\mathbf{g}) + (\mathbf{h})_{2\delta}$ is closed (as a subset of $\mathbb{R}[\mathbf{x}]_{2\delta}$ with the Euclidean topology), and therefore $\overline{\mathcal{Q}_{2\delta}(\mathbf{g})} \subset \mathcal{Q}_{2\delta}(\mathbf{g}) + (\mathbf{h})_{2\delta}$. Thus:

$$f = g + h = \sum_{i=0}^{s} s_i g_i + \sum_{i=1}^{m} p_i h_i \in \mathcal{Q}_{2\delta}(\mathbf{g}) + (\mathbf{h})_{2\delta},$$

where we set $g_0 = 1$ for notation convenience, $g = \sum_{i=0}^{s} s_i g_i \in \mathcal{Q}_{2\delta}(\mathbf{g})$ and $h = \sum_{i=1}^{m} p_i h_i \in (\mathbf{h})_{2\delta}$. It is then enough to prove that $\langle \Lambda^* | qg \rangle = \langle \Lambda^* | qh \rangle = 0$ where $\deg(qg) \leq b, \deg(qh) \leq b$ for $b = 2\delta + a - \deg(f) = 2\delta + 2\rho + 2D - \deg(f) - 2$.

We start by proving $\langle \Lambda^* | qh \rangle = 0$. We deduce from lemma 3.4.5 that for d big enough we have $(\mathbf{h})_b \subset \overline{\mathcal{Q}_{2d}(\mathbf{g})}$ and $\mathcal{L}_{2d}(\mathbf{g})^{[b]} \subset \mathcal{L}_b(\pm \mathbf{h})$. Therefore

$$\langle \Lambda^* | qh \rangle = \left\langle (\Lambda^*)^{[b]} \middle| qh \right\rangle = 0.$$

Now we prove that $\langle \Lambda^* | qg \rangle = 0$. Since $\delta + (a - \deg(f)) \leq d$, we can apply Lemma 3.5.3 with $g \in \mathcal{Q}_{2\delta}(\mathbf{g})$ and $t = a - \deg(f) \geq \deg(q)$, and conclude that $\langle \Lambda^* | qg \rangle = 0$, as desired.

Therefore $\langle \Lambda^* | qf \rangle = \langle \Lambda^* | qg \rangle + \langle \Lambda^* | qh \rangle = 0$ for all q of degree $\leq a - \deg(f)$ and (3.3) is satisfied. This implies that $(\Lambda^*)^{[a]} \in \mathcal{L}_{2d}(\mathbf{g}, \pm f)^{[a]}$. Therefore we can apply Theorem 3.4.20 to conclude that the flat truncation property is satisfied for Λ^*. □

Let us briefly comment the degree conditions in Theorem 3.5.4.

(i) If S has nonempty interior, it is not necessary to check the first condition, since in this case $\operatorname{supp} Q = 0$. More generally if the quadratic module is reduced, that is if $\sqrt[\mathbb{R}]{\operatorname{supp} Q} = \operatorname{supp} Q$, the first condition is automatically satisfied;

(ii) The second condition is the key one: it tells us that flat truncation happens when the ideal of the minimizers, truncated in the appropriate degree, can be described using the truncated quadratic module and the truncated ideal generated by $f - f^*$;

(iii) The third condition is technical, derived from Lemma 3.5.3. It allows to move from $\mathcal{L}_{2d}^{\min}(\mathbf{g})$ to $\mathcal{L}_{2d}^{(1)}(\mathbf{g}, \pm(f - f^*))$, where we can apply the results of the previous section.

We illustrate Theorem 3.5.4 in the following example, showing how it can help to predict the flat truncation degree.

Example 3.5.5. We continue Example 3.3.6. Notice that $f - f^* = x^2 \in Q_2 := \mathcal{Q}_2(\mathbf{g}) = \mathcal{Q}_2(1 - x^2 - y^2, x + y - 1)$ (i.e. the SoS hierarchy is exact) and then the moment hierarchy has finite convergence. Using Theorem 3.5.4, we analyze if flat truncation holds at some degree. We have $\mathcal{I}(S^{\min}) = (x, y-1) \subset \sqrt[\mathbb{R}]{\operatorname{supp}(Q + (f - f^*))} = \sqrt[\mathbb{R}]{\operatorname{supp}(Q + (x^2))}$ where $Q := \mathcal{Q}(1 - x^2 - y^2, x + y - 1)$. Indeed:

$$
\begin{aligned}
x &= \frac{x^2 + (y-1)^2}{2} + \frac{1 - x^2 - y^2}{2} + x + y - 1 \in Q_2 \subset \overline{Q_2 + (x^2)_2} \\
-x + \varepsilon &= \frac{\varepsilon}{2}\Big(1 - \frac{x^2}{\varepsilon^2} + (1 - \frac{x}{\varepsilon})^2\Big) \in Q_2 + (x^2)_2 \ \forall \varepsilon > 0 \Rightarrow -x \in \overline{Q_2 + (x^2)_2} \\
1 - y &= \frac{1}{2}\Big(x^2 + (1-y)^2 + 1 - x^2 - y^2\Big) \in Q_2 \subset \overline{Q_2 + (x^2)_2} \\
y - 1 &= x + y - 1 - x \in Q_2 + \overline{Q_2 + (x^2)_2} = \overline{Q_2 + (x^2)_2}
\end{aligned}
$$

that implies $(x, y-1)_1 \subset \operatorname{supp}(\overline{Q_2 + (x^2)_2}) \subset \sqrt[\mathbb{R}]{\operatorname{supp}(Q + (f - f^*))}$ and thus $\dim \frac{\mathbb{R}[\mathbf{x}]}{\operatorname{supp}(Q + (x^2))} = 0$. Theorem 3.5.4 implies that flat truncation holds for a high enough order d of the moment relaxation.

We investigate the degree conditions in Theorem 3.5.4 to prove that flat truncation happens for the moment relaxation at order $d = 1$. We have $I(S^{\min}) = (x, y-1)$, $\rho = 1$, $d_{\mathbf{g}} = 1$, $\deg(f) = 2$, $D = 1$ and $\delta = 1$.

(i) As S has nonempty interior, $\operatorname{supp} Q = 0$ and the first point (i) is satisfied.

(ii) Notice that $2(\rho - 1 + D) = 2$, and therefore we have to show that $(x, y-1)_2 \subset \overline{Q_2 + (x^2)_2}$. Since we have shown above that $(x, y-1)_1 \subset \overline{Q_2 + (x^2)_2}$, it is enough to prove that $\pm x^2, \pm x(y-1), \pm(y-1)^2 \in \overline{Q_2 + (x^2)_2}$. Now, $\pm x^2$, $(y-1)^2 \in \overline{Q_2 + (x^2)_2}$ by definition. Finally:

$$
\begin{aligned}
-(y-1)^2 &= 1 - y^2 - x^2 + x^2 + 2(x + y - 1) - 2x \in Q_2 + \overline{Q_2 + (x^2)_2} = \overline{Q_2 + (x^2)_2} \\
\pm x(y-1) &= \frac{1}{2}\Big((\pm x + (y-1))^2 - x^2 - (y-1)^2\Big) \in \overline{Q_2 + (x^2)_2},
\end{aligned}
$$

concluding the proof of the second point (ii).

(iii) We have $1 = \delta + 2\rho + 2D - \deg(f) - 2 \le d = 1$, and thus the third point (iii) is satisfied.

Therefore flat truncation happens at degree $\rho - 1 = 0$ for the moment relaxation at order $d = 1$.

Related properties have been previously investigated. It is shown in [Nie13b, th. 2.2] that, under genericity assumptions, if for an order d big enough we have $f^*_{\mathrm{SoS},d} = f^*_{\mathrm{Mom},d}$ (strong duality) and sup = max in the definition of $f^*_{\mathrm{SoS},d}$, then there is finite convergence (that is $f^*_{\mathrm{Mom},d} = f^*$) if and only if flat truncation is satisfied for every $\Lambda \in \mathcal{L}^{\min}_{2d}(\mathbf{g})$ (or, equivalently, if it is satisfied for $\Lambda^* \in \mathcal{L}^{\min}_{2d}(\mathbf{g})$ generic). Theorem 3.5.4 applies for different cases, for instance when there is finite convergence but the SoS hierarchy is not exact (see example 3.3.14). This is possible since our analysis investigates the closure of the quadratic modules we are considering. Furthermore, under genericity assumption, as a corollary of Theorem 3.5.4 we will show (in Theorem 3.5.7 and Corollary 3.5.9) that we have finite convergence, the SoS moment hierarchies are exact and the flat truncation property is satisfied.

Another improvement made is the estimation of the order d of the relaxation that is sufficient to have flat truncation, answering a question in [Nie13b]. To the best of out knowledge, this is is the first result in this direction. These conditions depends on properties of the minimizers and the quadratic module $\mathcal{Q}_{2d}(\mathbf{g})$ that might be difficult to check a priori. However they may be analyzed in some specific cases, such as optimization problems with a single minimizer, to deduce more precise bounds.

3.5.2 Boundary hessian conditions

In this section, we show that if regularity conditions, known as Boundary Hessian Conditions (BHC), are satisfied, then the flat truncation property holds. These are conditions on the minimizers of a polynomial f on a basic semialgebraic set S introduced by Marshall in [Mar06] and [Mar09], which are particular cases of the so called local-global principle. Under these conditions, global properties of polynomials (e.g. $f \in Q$) can be deduced from local properties (e.g. checking the BHC at the minimizers of f on $\mathcal{S}(Q)$). We refer to [Sch05a], [Sch06] and [Mar08, ch. 9] for more details. We introduce BHC conditions following [Nie14].

Definition 3.5.6 (Boundary Hessian Conditions)**.** Consider a POP with inequality constraints $\mathbf{g} = g_1,\dots,g_r$, equality constraints $\mathbf{h} = h_1,\dots,h_s$ and objective function f. Let $V = \mathcal{V}(\mathbf{h}) \subset \mathbb{R}^n$ and suppose that $Q = \mathcal{Q}(\mathbf{g}, \pm\mathbf{h})$ is Archimedean. We say that the *Boundary Hessian Conditions* hold at a minimizer point $\xi \in S(\mathbf{g}, \pm\mathbf{h})$ of f if ξ is a smooth point of V and:

(i) we can choose $g_{i_1} = t_1,\dots,g_{i_k} = t_k$ that are part of a regular system of parameters $t_1,\dots,t_m$, $m \geq k$, for V at ξ and for some neighbourhood U of ξ we have $\mathcal{S}(g_{i_1},\dots,g_{i_k},\pm\mathbf{h}) \cap U = \mathcal{S}(\mathbf{g}, \pm\mathbf{h}) \cap U$;

(ii) On V, locally at ξ we have that $\nabla f = a_1\nabla t_1 + \cdots + a_m\nabla t_m$, where a_i are strictly positive real numbers;

(iii) On V, locally at ξ we have that $hess(f)(0,\dots,0,t_{k+1},\dots t_m)$ is positive definite in $t_{k+1},\dots t_m$.

These conditions are related to standard conditions in optimization at a point $\xi \in S$ (see e.g. [Ber99]). Hereafter, the active constraints at $\xi \in S$ are the constraints $g_{i_1},\dots,g_{i_m}$ such that $g_{i_j}(\xi) = 0$ (see also Definition 2.3.2). To simplify the description of these conditions, we consider a constraint $\pm g(x) \geq 0$ as a single (equality) constraint. Therefore an equality constraint defining the set S is an active constraint at a point $\xi \in S$.

- *Constraint Qualification Condition* (CQC): for the active constraints $g_{i_1},\dots,g_{i_m}$ at ξ, the gradients $\nabla g_{i_1}(\xi),\dots,\nabla g_{i_m}(\xi)$ are linearly independent.
- *Strict Complementary Condition* (SCC): for the active constraints $g_{i_1},\dots,g_{i_m}$ at ξ, there exist $a_1,\dots,a_m \in \mathbb{R}$ with $a_j > 0$ if g_{i_j} is not an equality constraint such that $\boldsymbol{\nabla} f(\xi) = a_1\nabla g_{i_1}(\xi) + \cdots + a_m\nabla g_{i_m}(\xi)$.
- *Second Order Sufficiency Condition* (SOSC): for $L(x) = f(x) - \sum_{j=1}^{m} a_j g_{i_j}$ with $a_i > 0$ if $g_{i_j}(x)$ is not an equality constraint, we have $\forall v \in \langle \nabla g_{i_1}(\xi),\dots,\nabla g_{i_m}(\xi)\rangle^{\perp}$, $v \neq 0$, $v^t\nabla^2 L(\xi)v > 0$.

If these conditions are satisfied at every minimizer ξ, then the BHC conditions are satisfied with the active sign constraints at ξ as regular parameters $t_1 = g_{i_1},\dots,t_k = g_{i_k}$, see [Nie14].

Notice that when BHC hold, the minimizers are non-singular, isolated points and thus finite. It is proved in [Mar06] that if BHC holds at every minimizer of f on $\mathcal{S}(\mathbf{g})$ then $f - f^* \in \mathcal{Q}(\mathbf{g})$, which implies that the SoS hierarchy is exact. [Nie14] proved that the BHC at every minimizer of f, which hold generically, implies the SoS finite convergence property.

In this section, we prove that, if the BHC hold, then the flat truncation property holds and the moment hierarchy is exact.

Theorem 3.5.7. *Let $f \in \mathbb{R}[\mathbf{x}]$, $Q = \mathcal{Q}(\mathbf{g})$ be an Archimedean finitely generated quadratic module and assume that the BHC hold at every minimizer of f on $S = \mathcal{S}(\mathbf{g})$. Then:*

- *the SoS hierarchy is exact;*
- *the flat truncation holds for $\Lambda^* \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ generic, at relaxation order d satisfying conditions (i)-(iii) in Theorem 3.5.4;*
- *the moment hierarchy is exact.*

Proof. If BHC hold at every minimizer of f on $\mathcal{S}(\mathbf{g})$ then $S^{\min}$ is finite and$f - f^* \in \mathcal{Q}(\mathbf{g})$ (see [Mar06]), which implies that the SoS hierarchy is exact and thus the moment hierarchy has finite convergence. Moreover if the BHC conditions hold at every minimizer of f on S, then $\dim \frac{\mathbb{R}[\mathbf{x}]}{\operatorname{supp}(Q+(f-f^*))} = 0$ (see the proof of [Mar06, th. 2.3], where it is shown that the field of fractions of $\mathbb{R}[\mathbf{x}]$ modulo any minimal prime ideal lying over $\operatorname{supp}(Q + (f - f^*))$ is isomorphic to $\mathbb{R}$, that implies $\dim \frac{\mathbb{R}[\mathbf{x}]}{\operatorname{supp}(Q+(f-f^*))} = 0$). Then we conclude applying Theorem 3.5.1 and Theorem 3.5.4. □

Example 3.5.8. Consider $f = x$ and $g_1 = 1 - x^2 - y^2$, $g_2 = x + y - 1$ (this is a variation of Example 3.3.6 and Example 3.5.5). Let us show that CQC, SCC and SOSC are satisfied at $(0,1)$, the only minimizer of the problem.

- CQC: $\nabla g_1(0,1) = (0,-2)$ and $\nabla g_2(0,1) = (1,1)$ are linearly independent.
- SCC: $\nabla f(0,1) = (1,0)$ and $\nabla f(0,1) = \frac{1}{2}\nabla g_1(0,1) + 1\nabla g_2(0,1)$.
- SOSC: as $\langle \nabla g_1(0,1), \nabla g_2(0,1)\rangle = \mathbb{R}^2$ we have $\langle \nabla g_1(0,1), \nabla g_2(0,1)\rangle^{\perp} = \{(0,0)\}$ and the condition is trivially satisfied.

Therefore BHC hold. We show also explicitly how these conditions imply the BHC. We have:

(i) g_1 and g_2 are part of a regular system of parameters at $(0,1)$ as $\nabla g_1(0,1)$ and $\nabla g_2(0,1)$ are linearly independent, see [Sha13a];

(ii) we have already shown that $\nabla f(0,1) = \frac{1}{2}\nabla g_1(0,1) + 1\nabla g_2(0,1)$ and thus the second point is also satisfied;

(iii) as g_1 and g_2 are a complete system of regular parameters at $(0,1)$, the third condition is trivially satisfied.

From Theorem 3.5.7 we deduce that flat truncation is satisfied for any $\Lambda \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ for d big enough. In analogy with Example 3.5.5, $\mathcal{I}(S^{\min})_2 \subset \operatorname{supp} \overline{\mathcal{Q}_2(\mathbf{g}) + (f - f^*)_2}$ and flat truncation holds at degree 0.

We show now that flat truncation and moment exactness hold *generically*. For polynomials $f \in \mathbb{R}[\mathbf{x}]_d$ and $g_1 \in \mathbb{R}[\mathbf{x}]_{d_1}, \ldots, g_s \in \mathbb{R}[\mathbf{x}]_{d_s}$, we say that a property holds generically (or that the property holds for generic $f, g_1, \ldots, g_s$) if there exists finitely many nonzero polynomials $\phi_1, \ldots, \phi_l$ in the coefficients of polynomials in $\mathbb{R}[\mathbf{x}]_d$ and $\mathbb{R}[\mathbf{x}]_{d_1}, \ldots, \mathbb{R}[\mathbf{x}]_{d_s}$ such that, when $\phi_1(f, \mathbf{g}) \neq 0, \ldots, \phi_l(f, \mathbf{g}) \neq 0$, the property holds.

Corollary 3.5.9. *For $f \in \mathbb{R}[\mathbf{x}]_d$ and $g_1 \in \mathbb{R}[\mathbf{x}]_{d_1}, \ldots, g_s \in \mathbb{R}[\mathbf{x}]_{d_s}$ generic satisfying the Archimedean condition:*

- *the SoS hierarchy is exact;*
- *the flat truncation holds for $\Lambda^* \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ generic, at relaxation order d satisfying conditions (i)-(iii) in Theorem 3.5.4;*
- *the moment hierarchy is exact.*

Proof. By [Nie14, th. 1.2] BHC hold generically. We apply Theorem 3.5.7 to conclude. □

Here is an example where BHC holds.

Example 3.5.10 (Robinson form)**.** We find the minimizers of Robinson form $f = x^6 + y^6 + z^6 + 3x^2y^2z^2 - x^4(y^2+z^2) - y^4(x^2+z^2) - z^4(x^2+y^2)$ on the unit sphere $h = x^2+y^2+z^2-1$. The Robinson polynomial has minimum $f^* = 0$ on the unit sphere, and the minimizers on $\mathcal{V}_{\mathbb{R}}(h)$ are:

$$\frac{\sqrt{3}}{3}(\pm1,\pm1,\pm1), \frac{\sqrt{2}}{2}(0,\pm1,\pm1), \frac{\sqrt{2}}{2}(\pm1,0,\pm1), \frac{\sqrt{2}}{2}(\pm1,\pm1,0).$$

BHC are satisfied at every minimizer (see [Nie14, ex. 3.2]), flat truncation holds and we can recover the minimizers from Theorem 3.5.7. We estimate the bounds of Theorem 3.5.4 and compare with the numerical experiments. It is not necessary to check (i), since $(h) = \sqrt[\mathbb{R}]{\operatorname{supp} \mathcal{Q}(\pm h)}$. For the point (ii), we estimate the regularity of the minimizers as the regularity of twenty generic points on a sphere, that is $\rho = 5$. Then $2\rho + 2D - 2 = 14$, and thus we expect flat truncation for $d \geq 7$. For the point (iii), we need to have $d \geq \delta + 2\rho + 2D - \deg(f) - 2 \geq 3+10+6-6-2 = 11$. However, in practice for this example we have flat truncation numerically at order 6 and not before (using the SDP solver SDPA). We recover a good approximation of the minimizers at this order:

```
v, M = minimize(f, [h], [], X, 6)
w, Xi = get_measure(M)
```

Here $f^*_{\mathrm{Mom},6} \approx v = -1.27211 \cdot 10^{-7}$ and the minimizers with positive coordinates are (all the twenty minimizers are found):

	ξ_1	ξ_2	ξ_3	ξ_4
x	0.577351068999	$8.812477930640\,10^{-12}$	0.707107158043	0.707107157553
y	0.577351069076	0.707107158048	$1.271729446125\,10^{-13}$	0.707107157555
z	0.577351066102	0.707107158048	0.707107158042	$2.478771201340\,10^{-9}$

3.5.3 Finite semialgebraic sets

In this section we consider the case when $S = \mathcal{S}(\mathbf{g}) = \{\xi_1, \dots, \xi_r\} \subset \mathbb{R}^n$ is non-empty and finite.

Theorem 3.5.11. *Let $Q = \mathcal{Q}(\mathbf{g})$ and suppose that* $\dim \frac{\mathbb{R}[\mathbf{x}]}{\operatorname{supp} Q} = 0$. *Then:*

- *S is finite;*
- *the flat truncation holds for $\Lambda^* \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ generic, at relaxation order d big enough satisfying conditions (i)-(iii) in Theorem 3.5.4;*
- *the moment hierarchy is exact.*

Proof. Since $\dim \frac{\mathbb{R}[\mathbf{x}]}{\operatorname{supp} Q} = 0$, we deduce that S is finite and we have moment finite convergence from Theorem 3.4.20, Theorem 3.4.19 and Theorem 3.5.1. Indeed if d is big enough then flat truncation is satisfied for any $\Lambda \in \mathcal{L}_{2d}(\mathbf{g})$, in particular for $\Lambda^* \in \mathcal{L}_{2d}^{\min}(\mathbf{g})$ generic. We conclude applying Theorem 3.5.4. □

As corollaries, we see that the conclusions of Theorem 3.5.11 hold:

- for the moment hierarchy $\mathcal{L}_{2d}(\Pi\mathbf{g})$ when $S = \mathcal{S}(\mathbf{g}) = \mathcal{S}(\Pi\mathbf{g})$ is finite, since by the real Nullstellensatz,

$$\dim \frac{\mathbb{R}[\mathbf{x}]}{\operatorname{supp} Q(\Pi\mathbf{g})} = \dim \frac{\mathbb{R}[\mathbf{x}]}{\operatorname{supp} \mathcal{O}(\mathbf{g})} = \dim \frac{\mathbb{R}[\mathbf{x}]}{\sqrt{\operatorname{supp} \mathcal{O}(\mathbf{g})}} = \dim \frac{\mathbb{R}[\mathbf{x}]}{\mathcal{I}(\mathcal{S}(\mathbf{g}))} = 0.$$

  See [Nie13c, th. 4.1] and [LLR08, rem. 4.9].

- for the moment hierarchy $\mathcal{L}_d(\mathbf{g}, \pm\mathbf{h})$ when $\mathcal{V}_{\mathbb{R}}(\mathbf{h})$ is finite, since for $Q = Q(\mathbf{g}, \pm\mathbf{h})$,

$$\dim \frac{\mathbb{R}[\mathbf{x}]}{\operatorname{supp} Q} = \dim \frac{\mathbb{R}[\mathbf{x}]}{\sqrt{\operatorname{supp} Q}} = \dim \frac{\mathbb{R}[\mathbf{x}]}{\sqrt[\mathbb{R}]{\operatorname{supp} Q}} \leq \dim \frac{\mathbb{R}[\mathbf{x}]}{\sqrt[\mathbb{R}]{(\mathbf{h})}} = 0.$$

  See [Nie13c, th. 1.1] and [LLR08]. This includes Polynomial Optimization problems with binary variables and equations of the form $x_i^2 - x_i = 0$, for which moment relaxations are of particular interest, see e.g. [Lau03].

However, Theorem 3.5.11 is more general than the results above, see for instance Example 3.4.21

Notice that, even if the SoS hierarchy has the finite convergence property and the moment hierarchy is exact, it may not be SoS exact for a finite real variety, as shown in Example 3.3.13 and Example 3.3.14.

Example 3.5.12 (Gradient ideal)**.** We compute the minimizers of Example 3.3.14. Let $f = (x^4y^2 + x^2y^4 + z^6 - 2x^2y^2z^2) + x^8 + y^8 + z^8 \in \mathbb{R}[x,y,z]$. We want to minimize f over the gradient variety $\mathcal{V}_{\mathbb{R}}\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ with $\dim \frac{\mathbb{R}[\mathbf{x}]}{(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})} = 0$. By Theorem 3.5.11, we deduce that flat truncation holds for an order of relaxation d high enough. In this example, we have $\rho = 1$, $D = 4$, $\deg(f) = 8$, $\delta \geq 4$, so that we expect flat truncation at an order $d \geq 4$, from Theorem 3.5.4.

```
v, M = minimize(f, differentiate(f,X), [], X, 4)
w, Xi = get_measure(M, 2.e-2)
```

The approximation of the minimum $f^* = 0$ is $v = -1.6279 \cdot 10^{-9}$, and the decomposition with a threshold of $2 \cdot 10^{-2}$ gives the following numerical approximation of the minimizer (the origin):

$$\xi = (2.976731510689691\ 10^{-17}; -9.515032317137384\ 10^{-19}; 3.763401209219283\ 10^{-18}).$$

3.5.4 Gradient, KKT and polar ideals

Another approach which has been investigated to make the hierarchies exact, is to add equality constraints satisfied by the minimizers (and independent of the minimum f^*) to a Polynomial Optimization Program.

For global optimization we can consider the gradient equations (see [NDS06]): obviously $\nabla f(x^*) = \mathbf{0}$ for all the minimizers x^* of f on $S = \mathbb{R}^n$. For constrained optimization we can consider Karush–Kuhn–Tucker (KKT) constraints, adding new variables (see [DNP07]) or projecting them to the variables $\mathbf{x}$ (Jacobian equations, see [Nie13a]). We shortly describe them.

Let $g_1, \ldots, g_r, h_1, \ldots, h_s \in \mathbb{R}[\mathbf{x}]$ defining $S = \mathcal{S}(\mathbf{g}, \pm\mathbf{h})$, and let $f \in \mathbb{R}[\mathbf{x}]$ be the objective function. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$ and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_s)$ be variables representing the *Lagrange multipliers* associated with $\mathbf{g}$ and $\mathbf{h}$. The *KKT constraints* associated to the optimization problem $\min f(x) \colon x \in \mathcal{S}(\mathbf{g}, \pm\mathbf{h})$ are:

$$\begin{cases} \dfrac{\partial f}{\partial x_i} - \displaystyle\sum_{k=1}^{r} \lambda_k^2 \frac{\partial g_k}{\partial x_i} - \sum_{j=1}^{s} \gamma_j \frac{\partial h_j}{\partial x_i} = 0 & \forall i \\ \Lambda_k g_k = 0, \quad h_j = 0, \quad g_k \geq 0 & \forall j, k, \end{cases} \tag{3.4}$$

where the polynomials belong to $\mathbb{R}[\mathbf{x}, \boldsymbol{\gamma}, \boldsymbol{\lambda}]$. These are sufficient but not necessary conditions for $x^* \in S$ being a minimizer.

Let $x^* \in S$ and $g_{i_1}, \ldots g_{i_k}$ be the active constraints at x^*. The KKT constraints are necessary if the Constraint Qualification Condition (CQC) holds, that is, if $\nabla h_1(x^*), \ldots, \nabla h_s(x^*), \nabla g_{i_1}(x^*), \ldots, \nabla g_{i_k}(x^*)$ are linearly independent at the minimizer $x^* \in S$ (also called Linear Independence Constraint Qualification in [NW06, th. 12.1]). We cannot avoid the CQC hypothesis: for example if $f = x_1 \in \mathbb{R}[x_1]$ and $g_1 = x_1^3 \in \mathbb{R}[x_1]$, then $x^* = 0$ is a minimizer, but the KKT equations are not satisfied at $x^* = 0$. To avoid this problem we define the *polar ideal*. Observe from eq. (3.4) that, if KKT constraints are satisfied at x and

- if g_i is not an active constraint at x, then $\lambda_i = 0$;
- if $g_{i_1}, \ldots g_{i_k}$ are the active constraints at x, then the gradients $\nabla f(x), \nabla h_1(x), \ldots, \nabla h_s(x), \nabla g_{i_1}(x), \ldots, \nabla g_{i_k}(x)$ are linearly dependent.

Definition 3.5.13. For $f, g_1, \ldots, g_r, h_1, \ldots, h_s \in \mathbb{R}[\mathbf{x}]$ as before, the *polar ideal* is defined as follows:

$$J := (\mathbf{h}) + \sum_{A = \{a_1, \ldots, a_k\} \subset \{1, \ldots r\}} \Big(\operatorname{rank} \operatorname{Jac}(f, \mathbf{h}, g_{a_1}, \ldots, g_{a_k}) < s + k + 1\Big) \prod_{b \notin A} g_b.$$

where $\left(\operatorname{rank}\operatorname{Jac}(f,\mathbf{h},g_{a_1},\dots,g_{a_k})<l\right)$ is the ideal generated by the $l\times l$ minors of the Jacobian matrix $\operatorname{Jac}(f,\mathbf{h},g_{a_1},\dots,g_{a_k})$.

We could replace the generators of the ideal in this definition by polynomials defining the same variety. This variety, known also as Jacobian or augmented Jacobian variety, coincides with the one defined by $h_1,\dots,h_{m_1},\varphi_i,\dots,\varphi_r$ in [Nie13a].

The improvement that we make from the KKT constraints is to consider conditions that are necessary for being a minimizer, similar to Fritz John Optimality Conditions (see [Ber99, sec. 3.3.5]). Indeed we prove in the next lemma that every minimizer belongs to $\mathcal{V}_{\mathbb{R}}(J)$.

Lemma 3.5.14. *Let x^* be a minimizer of f on $S=\mathcal{S}(\mathbf{g},\pm\mathbf{h})$. Then $x^*\in\mathcal{V}_{\mathbb{R}}(J)$.*

Proof. Since $x^*\in S$, then $x^*\in\mathcal{V}_{\mathbb{R}}(\mathbf{h})$.

If the CQC hold at x^*, then x^* is a KKT point (see [NW06, th. 12.1]) and $\nabla f(x)=\sum_j\gamma_j\nabla\mathbf{h}_j(x)+\sum_j\lambda_j^2\nabla\mathbf{g}_j(x)$ for some γ_j and λ_i in $\mathbb{R}$. As $\lambda_k=0$ if g_k is not an active constraint, we have that

$$\nabla f(x^*),\nabla h_1(x^*),\dots,\nabla h_r(x^*),\nabla g_{i_1}({}^*x),\dots,\nabla g_{i_k}(x^*)$$

are linearly dependent, where $g_{i_1},\dots g_{i_k}$ are the active constraints at x^*. Thus

$$\operatorname{rank}\operatorname{Jac}(f(x^*),\mathbf{h}(x^*),g_{a_1}(x^*),\dots,g_{a_k}(x^*))<s+k+1\text{ if }\{i_1,\dots,i_k\}\subset\{a_1,\dots,a_k\}.$$

On the other hand, if $i_j\notin\{a_1,\dots,a_k\}$ then $g_{i_j}(x^*)=0$. This implies $x^*\in\mathcal{V}_{\mathbb{R}}(J)$.

If the CQC do not hold at x^* and $g_{i_1},\dots,g_{i_k}$ are the active constraints, then the gradients $\nabla h_1(x^*),\dots,\nabla h_s(x^*)$ and $\nabla g_{i_1}(x^*),\dots,\nabla g_{i_k}(x^*)$ are linearly dependent. This implies that $\nabla f(x^*)$, $\nabla h_1(x^*),\dots,\nabla h_s(x^*)$ and $\nabla g_{i_1}(x^*),\dots,\nabla g_{i_k}(x^*)$ are also linearly dependent, and we conclude as in the previous case. □

Theorem 3.5.15. *Let $Q=\mathcal{Q}(\mathbf{g},\pm\mathbf{h})$ and $J=(\mathbf{h}')$ be the polar ideal, where $\mathbf{h}'$ is a finite set of generators of J. If $\dim\frac{\mathbb{R}[\mathbf{x}]}{\operatorname{supp}(\mathcal{Q}(\mathbf{g})+(\mathbf{h}'))}=0$, then:*

- *the flat truncation holds for $\Lambda^*\in\mathcal{L}_{2d}^{\min}(\mathbf{g},\pm\mathbf{h}')$ generic, at relaxation order d is big enough;*
- *the moment hierarchy $(\mathcal{L}_{2d}(\mathbf{g},\pm\mathbf{h}'))_{d\in\mathbb{N}}$ is exact.*

In particular this holds when $\mathcal{V}_{\mathbb{R}}(J)$ is finite.

Proof. Minimizers belongs to $\mathcal{V}_{\mathbb{R}}(J)$ by Lemma 3.5.14. Then moment exactness follows from Theorem 3.5.11. □

The assumption in [NDS06], [DNP07] and [Nie13a] for finite convergence and SoS exactness are smoothness conditions or radicality assumptions on the associated complex variety. In particular, Assumption 2.2 in [Nie13a, th. 2.3] requires the varieties defined by the active constraints to be non-singular to conclude finite convergence of the hierarchy. Our condition for finite convergence and flat truncation is of a different nature, since it is on the finiteness of the real polar variety. For instance we can apply Theorem 3.5.15 in Example 3.5.16, but Assumption 2.2 in [Nie13a] is not satisfied, since the minimizer is a singular point. Moreover notice that in our theorem we use only the defining inequalities $\mathbf{g}$ and not their products $\Pi\mathbf{g}$, as done in [Nie13a, th. 2.3] (in other words, we only need the quadratic module and not the preordering).

In the following example, BHC are not satisfied. But adding the polar constraints, we obtain an exact relaxation with the flat truncation property.

Example 3.5.16 (Singular minimizer)**.** We minimize $f = x$ on the compact semialgebraic set $S = \mathcal{S}(x^3 - y^2, 1 - x^2 - y^2)$. We have $f^* = 0$ and the only minimizer is the origin, which is a singular point of the boundary of S. Thus BHC do not hold, and we cannot apply Theorem 3.5.7. We have $\dim \frac{\mathbb{R}[\mathbf{x}]}{\mathrm{supp}(Q+(x))} = 0$ since $\mathrm{supp}(Q + (x)) \supset (x, y^2)$, but we cannot apply Theorem 3.5.4, as we don't have finite convergence of the SoS and moment hierarchies. Indeed $x \notin Q = \mathcal{Q}(x^3 - y^2, 1 - x^2 - y^2)$, since $x \notin \mathcal{Q}(x^3, 1 - x^2)$. This implies that the SoS and moment hierarchies do not have finite convergence, following Example 3.3.9. This example also shows that we cannot remove the hypothesis of moment finite convergence in Theorem 3.5.4.

To get flat truncation, we add the polar equations, that define a finite real polar variety, as we show in the following. First notice that, since $\mathcal{V}(x^3 - y^2)$ is singular, Assumption 2.2 in [Nie13a] is not satisfied and the finite convergence of the relaxation $\mathcal{O}_{2d}(\mathbf{g}, \pm\mathbf{h}')$ using the polar variety cannot be deduced from [Nie13a, th. 2.3]. The generators of the polar variety are $\mathbf{h}' = (1 - x^2 - y^2)(x^3 - y^2)$, $y(1 - x^2 - y^2)$, $y(x^3 - y^2)$. The real roots are $(-1, 0)$, $(1, 0)$, $(0, 0)$ and the two real intersections of $1 - x^2 - y^2 = 0$ and $x^3 - y^2 = 0$. Therefore $\dim \frac{\mathbb{R}[\mathbf{x}]}{\mathrm{supp}(Q+(\mathbf{h}'))} \leq \dim \frac{\mathbb{R}[\mathbf{x}]}{\sqrt[\mathbb{R}]{(\mathbf{h}')}} = 0$, and Theorem 3.5.15 implies flat truncation. We recover the minimizer considering the moment relaxation of order 5:

```
v, M = polar_minimize(f, [], [x^3-y^2,1-x^2-y^2], X, 5)
w, Xi = get_measure(M, 2.e-3)
```

The approximation of the minimum $f^* = 0$ is $v = -0.0045$, and the decomposition of the moment sequence with a threshold of $2 \cdot 10^{-3}$ gives the following approximation of the minimizer (the origin):

$$\xi = (-0.004514367348787526, 2.1341684460860045\ 10^{-21}).$$

The error of approximation on ξ is of the same order than the error on the minimum f^*.

3.6 Example: optimal power flow

In this section, we show a concrete application of the moment hierarchy (in particular, of flat truncation and the extraction of minimizers). We investigate instances of the so-called *Alternating Current-Optimal Power Flow* problem (AC-OPF), see Section 3.6, and we present some computations combining semidefinite interior point solvers and a local refinement step in Section 3.6.1.

Probelm: size of the semidefinite program

A known issue of the sum of squares and moment hierarchies is the size of the underlying semidefinite programs. Indeed, if we look at the order d moment relaxation in Equation (1.9), the positive semidefinite constraints:

$$H_\Lambda^d \succcurlyeq 0, H_{g_1 \star \Lambda}^{N_1} \succcurlyeq 0, \dots, H_{g_r \star \Lambda}^{N_r} \succcurlyeq 0$$

are defined by matrices of dimensions:

$$\binom{n+d}{d} \times \binom{n+d}{d}, \binom{n+N_1}{N_1} \times \binom{n+N_1}{N_1}, \dots, \binom{n+N_r}{N_r} \times \binom{n+N_r}{N_r}.$$

Due to the rapidly increasing size of the matrices, the solution of the moment relaxation becomes quickly intractable as the number of variables *n* becomes large. And this is precisely the situation in many real-life optimization problem, including the AC-OPF problem that we are going to study.

To overcome this limitation, a natural solution is to exploit the structure of the problem, and in particular its *sparsity* structure, in order to reduce the dimension of the matrices.

The technique that we are going to use to exploit the sparsity is the correlative-term sparsity, implemented in the Julia package `TSSOS.jl`[2], introduced by Wang, Magron and Lasserre [WML20; MW21]. Roughly speaking, this consists of two steps:

- first, there is a *correlative sparsity* step, where the variables are partitioned in different subsets according to the support of the objective function and of the defying inequalities;
- second, there is a *term sparsity* step for every subset of variables, where block diagonal moment and localizing matrices are constructed in an iterative manner.

Alternating current-optimal power flow

The Alternate current- optimal power flow (AC-OPF) is an important problem in power systems. This problem can be formulated as a polynomial optimization problem with either real variables or complex variables, and recently has been the center of an important research effort: many strategies have been developed to tackle this problem, and different test cases have been investigated to verify the performance of these strategies. We refer to [Bab+21] and references therein for a precise formulation of the problem and a list of test cases.

The AC-OPF problems are challenging, nonlinear, nonconvex problems, and nonlinear programming tools can usually produce a locally optimal solution that might differ for the global one. In particular, local solutions give *upper bounds* for the true minimum. It is then possible to use Lasserre's hierarchies to produce *lower bounds* for the minimum: if the lower bound and the upper bound are equal (or their difference is within a certain tolerance) then we can certify global optimality.

In the next examples, we develop this idea in some examples.

3.6.1 Examples

We study Pan European Grid Advanced Simulation and State Estimation (PEGASE) test cases. In particular, we consider `pglib_opf_case89_pegase` and `pglib_opf_case89_pegase__api`, see [Bab+21]. This test cases are available in the benchmark library Power Grid Lib - Optimal Power Flow[3], and also in the POEMA database[4] of polynomial optimization problems.

We use `pglib_opf_case89_pegase` to describe our procedure, and summarize our results in Table 3.2. We first verify the results in [Bab+21] (first column) using the `KNITRO`[5]. This gives an upper bound on the minimum of the problem.

Then, we use `TSSOS` to compute a lower bound for the problem, using the minimal initial relaxation step (see [WML22]), showed in the second column. In this computation, we impose

[2] https://github.com/wangjie212/TSSOS

[3] https://github.com/power-grid-lib/pglib-opf

[4] https://github.com/PolynomialMomentOptimization/data

[5] https://www.artelys.com/solvers/knitro/

Table 3.2: AC-OPF PEGASE case89

Test case	[Bab+21]	KNITRO	TSSOS	gap	Mom. start
pglib_opf_case89_pegase	1.0729e+05	1.0729e+05	1.0671e+05	0.005	1.0672e+05
pglib_opf_case89_pegase__api	1.3017e+05	1.3017e+5	1.0186e+05	0.217	1.2610e+05

a positivity constraint in the first variable: heuristically, this choice breaks the symmetry of the problem, and we obtain a unique minimizer that can be extracted from the order one moment matrix. The relative optimality gap obtained $(1.0729e+05+1.0671e+05)/1.0729e+05 \approx 0.005$ gives an upper bound on the error of the computations. A zero bound would certify that the local minimum computed from the solver is a global minimum.

We then exploit properties of the moment relaxation to improve the computations. We extract a point from the order one moment matrix (which is approximately flat), that is a good approximation of the global minimum, and we use it as starting point for KNITRO. In this way, the computation is improved (last column), and we obtain a smaller optimality gap as result.

Following this procedure, we are able to improve the optimal values in [Bab+21].

3.7 Summary and perspectives

Before suggesting some open questions and possible application, let us briefly summarize the content of the chapter.

We investigated the convex cones $\mathcal{L}_d(\mathbf{g})$ dual to the truncated quadratic modules $\mathcal{Q}_d(\mathbf{g})$ from a new perspective. We studied the kernels of moment matrices or annihilators of moment sequences in these cones and characterize the ideal they generate (Theorem 3.4.11). We focused on the zero dimensional case and its relationships with the flat truncation property (Theorem 3.4.19 and Theorem 3.4.20), that can be used to certify that a linear functional is coming from a measure.

The main contributions of the chapter are the applications of the previous analysis to flat truncation in Lasserre's moment hierarchy for Polynomial Optimization. We studied the flat truncation property in this context (Theorem 3.5.1) and deduced new necessary and sufficient conditions for flat truncation (Theorem 3.5.4). These conditions can be used to show that, under regularity and thus genericity assumptions (Boundary Hessian Conditions), the flat truncation property is satisfied (Theorem 3.5.7, Corollary 3.5.9). We applied these results to Polynomial Optimization on finite sets (Theorem 3.5.11) and for singular cases, adding polar equations, to obtain flat truncation (Theorem 3.5.15).

Theorem 3.5.4 provides the first known degree bounds for the flat truncation property to hold, in terms of the inequalities $\mathbf{g}$ and the objective function f (in particular depending on the regularity of the minimizers). An interesting research direction would be to investigate if it is possible to improve and clarify these degree bounds, for instance for optimization problem with a unique minimizer, or for special classes of problems (e.g. in graph theory, optimal power flow problems, ...).

Another possible research direction is to investigate regularity conditions, simpler than Boundary Hessian Conditions, that imply flat truncation for the moment relaxation of a certain order d.

Finally, the analysis of $\mathcal{L}_d(\mathbf{g})$ could be used to investigate the problem of strong duality in polynomial optimization problems, in two cases:

- when we have an Archimedean quadratic module, but the ball constraint is not explicit (using Lemma 2.5.8);
- when the quadratic module is not reduced (that is, when the support is not real radical), using Theorem 3.4.3 to find examples of non-closed truncated quadratic modules, and therefore examples of non-zero duality gap.

Chapter 4

Real Radical Computation

This chapter is based on [BM21].

4.1 Context and results

The solution of systems of polynomial equations has always been central in mathematics. The computation of these solutions can be tackled from symbolic point of view, where among the most important tools we find Groebner bases, resultants and the simultaneous diagonalization of multiplication matrices (see e.g. [EM07; CLO15]). More recently, symbolic-numeric methods have been proposed, with the aim to combine the speed of numerical computations and the robustness of exact computations, using for instance homotopy continuation or border bases techniques, see e.g. [SW05; MT12].

In these computations, it is important to specify the field where our solutions should live. On one hand, if we are interest on *complex* roots (or, more generally, solutions over an algebraically closed field), solving polynomial systems means to find equations for the radical of the ideal defined by the polynomials. On the other hand, if we are interested only in *real* roots (or, more generally, solutions over a real closed field) solving the polynomial system means to find equations for the *real radical* of the ideal defined by the polynomials.

In many real-world problems which can be modeled by polynomial constraints, real solutions are generally analyzed with particular attention, and finding equations vanishing on the real solutions without computing all the complex roots is a challenging question. This means that the computation of the vanishing ideal of the real solutions of an ideal I, that is, its real radical $\sqrt[\mathbb{R}]{I}$, is of particular interest.

For the computation of the real radical in the case of finitely many real solutions, a new symbolic-numeric algorithm has been proposed in [LLR08] and [Las+13]. This algorithm exploits properties of positive linear functionals, and is effectively performed solving a hierarchy of semidefinite programs, that compute at every step a positive truncated pseudo-moment sequence. For degree big enough, the annihilator of this sequence generates the real radical, and this condition is detected using the flat truncation property. Although this algorithm solves the problem in the zero dimensional case, finding a stopping criterion to certify that the equations of the real radical have been computed at a certain order is still an open question.

In this chapter, we propose a new stopping criterion, that applies in the positive dimensional case. More precisely, we present a new algorithm to compute the real radical of an ideal I (and, more generally, the S-radical of I), which is based on the idea above. A generic truncated positive linear functional Λ, that lies in the orthogonal of I, is computed solving a Moment Optimization Problem (MOP) (i.e. a semidefinite program). We show that, for a large enough degree of truncation, the annihilator of Λ generates the real radical of I, as in the zero dimensional case. We give an effective, general stopping criterion on the degree to detect when the prime ideals lying over the annihilator are real, and we compute the real radical as the intersection of real prime ideals lying over I. The final algorithm is described in Algorithm 4.5.1.

The method involves several ingredients, that exploit the properties of generic positive moment sequences. A new efficient algorithm is proposed to compute a graded basis of the annihilator of a truncated positive linear functional (Algorithm 4.3.1). We then propose a new algorithm to check that an irreducible decomposition of an algebraic variety is real, using a generic real projection to reduce to the hypersurface case (Algorithm 4.4.1). There we apply the Sign Changing Criterion, effectively performed with another exact MOP.

This criterion is always satisfied for a large enough degree of truncation, and it certifies that the annihilator generates the real radical if the generated ideal has no embedded components. An interesting feature of the approach is that it does not involve the computation complex solutions, which are not on a real component of the algebraic variety $\mathcal{V}(I)$.

4.1.1 Related works

Several approaches have been proposed to compute the real radical. Some of these methods are reducing to univariate problems [BN93; Neu98; BS99; Spa08], or exploiting quantifier elimination techniques [GV95], or using infinitesimals [RV02] or triangular sets and regular chains [XY02; Che+13].

Sums-of-Squares convex optimization and moment matrices are used in [LLR08; Las+13] to compute real radicals, when the set of real solutions is finite. Some properties of ideals associated to semidefinite programming relaxations are analyzed in [STW13], involving the simple point criterion. In [MWZ16] a stopping criterion is presented to verify that a Pommaret basis has been computed from the kernels of moment matrices involved in Sum of Squares relaxation. In [BHL16], a test based on sum-of-square decomposition is proposed to verify that polynomials vanishing on a subset of the semi-algebraic set are in the real radical.

In [SEDYZ21], an algorithm based on rational representations of equidimensional components of algebraic varieties and singular locus recursion is presented and its complexity is analyzed.

Our approach follows the algorithm in [LLR08; Las+13], which applies for zero-dimensional real ideals and uses the flat extension property (see e.g. [CF98; LM09]) as a stopping criterion: if the flat extension property holds then the annihilator of Λ generates the real radical of I, and this criterion is satisfied for a degree big enough.

But the question of finding an effective stopping criterion remained open for positive-dimensional real varieties (see e.g. [LR12, § 4.3]). In this work, we handle more specifically the positive-dimensional case. This case has also been analyzed in [MWZ16], where a stopping criterion is proposed to detect when a Pommaret basis has been computed. This test is generically satisfied for a large enough degree of truncation, but it does not certify that the basis generates the real radical.

4.1.2 Structure of the chapter

The chapter is structured as follows.

- Section 4.1 introduces the problem and describes the contributions of the chapter. In Section 4.1.1 we compare our results with the existing literature, and in Section 4.1.2 we describe the structure of the chapter.

- Section 4.2 present the main definitions and constructions that we will use through the chapter. In particular, in Section 4.2.1 we describe the idea behind the main algorithm, and in Section 4.2.2 we summarize the notions of numerical algebraic geometry that we will need.

- Section 4.3 is devoted to the computation of the basis the annihilator of a truncated positive linear functional (or, in other words, for the kernel of the moment matrix), and a new algorithm is presented.

- Section 4.4 presents an algorithm to detect if a real irreducible variety is defined over $\mathbb{R}$ and has dense real points. We call the definition of genericity (Section 4.4.1) and geometrical conditions for real radicality in Section 4.4.2. Finally, in Section 4.4.3 we describe our algorithm and provide the theoretical justification for it, and show its effectiveness in Section 4.4.4.

- Section 4.5 describes the main algorithm, that uses as subroutines the previous algorithms for the basis of the annihilator and to check if the real points are dense.

- Section 4.6 illustrates the behavior of the algorithm in different examples.

- Section 4.7 concludes the chapter, suggesting possible improvements of the algorithm.

4.2 Basic setting

For the purpose of the chapter, it is convenient to describe explicitly the equations defining a (basic, closed) semialgebraic set:

$$S = \mathcal{S}(\pm\mathbf{f},\mathbf{g}) = \{\xi \in \mathbb{R}^n \mid f_1(\xi) = 0,\dots, f_s(\xi) = 0, g_1(\xi) \geq 0,\dots, g_r(\xi) \geq 0\}.$$

We want to find an effective way to compute equations for:

$$\mathcal{I}_{\mathbb{R}}(\mathcal{S}(\pm\mathbf{f},\mathbf{g})) = \sqrt[\mathbb{R}]{\operatorname{supp}\mathcal{O}(\pm\mathbf{f},\mathbf{g})} = \{p \in \mathbb{R}[\mathbf{x}] \mid \exists m \in \mathbb{N},\ -p^{2m} \in (\mathbf{f}) + \mathcal{O}(\mathbf{g})\}.$$

In this context, this ideal is sometimes called the S-radical of $I = (\mathbf{f})$ and denoted $\sqrt[S]{I} = \mathcal{I}_{\mathbb{R}}(\mathcal{S}(\pm\mathbf{f},\mathbf{g}))$, e.g. in [SEDYZ21]. We are going to follow this convention in this chapter.

The S-radical $\sqrt[S]{I}$ is related to the real radical of an extended ideal I_S defined by introducing slack variables $t_1,\dots,t_r$ for each non-negativity constraint defining S: $I_S = (f_1,\dots,f_s,g_1 - t_1^2,\dots,g_r - t_r^2) \subset \mathbb{R}[x_1,\dots,x_n,t_1,\dots,t_r]$. Namely, we have $\sqrt[S]{I} = \sqrt[\mathbb{R}]{I_S} \cap \mathbb{R}[\mathbf{x}]$. Indeed, using the fact that taking radicals commutes with contraction:

$$\begin{aligned}\sqrt[\mathbb{R}]{I_S} \cap \mathbb{R}[\mathbf{x}] &= \sqrt[\mathbb{R}]{\operatorname{supp}(I_S + \Sigma\mathbb{R}[\mathbf{x},\mathbf{t}]^2)} \cap \mathbb{R}[\mathbf{x}] \\ &= \sqrt{(I_S + \Sigma\mathbb{R}[\mathbf{x},\mathbf{t}]^2) \cap -(I_S + \Sigma\mathbb{R}[\mathbf{x},\mathbf{t}]^2) \cap \mathbb{R}[\mathbf{x}]} \\ &= \sqrt{(I_S + \Sigma\mathbb{R}[\mathbf{x},\mathbf{t}]^2) \cap -(I_S + \Sigma\mathbb{R}[\mathbf{x},\mathbf{t}]^2) \cap \mathbb{R}[\mathbf{x}]} \\ &= \sqrt[\mathbb{R}]{I_S \cap \mathbb{R}[\mathbf{x}]}\end{aligned}$$

See also the remarks after Definition 1.1.12 and Lemma 1.1.24. Therefore, the ideal $\sqrt[\mathbb{R}]{I_S} \cap \mathbb{R}[\mathbf{x}]$ is real radical, and to show that $\sqrt[S]{I} = \sqrt[\mathbb{R}]{I_S} \cap \mathbb{R}[\mathbf{x}]$ we only need to prove that the associated real varieties are equal. But this follows easily, since the natural embedding $\mathbb{R}[\mathbf{x}] \subset \mathbb{R}[\mathbf{x},\mathbf{t}]$, giving the contraction $\cap\mathbb{R}[\mathbf{x}]$, correspond to the projection $\pi\colon \mathbb{R}^{n+r} \to \mathbb{R}^n$ and $S = \mathcal{S}(\pm\mathbf{f},\mathbf{g}) = \pi(\mathcal{V}_{\mathbb{R}}(I_S))$.

Therefore, we can reduce the computation of S-radicals to the computation of real radicals. In the following we will then focus on the computation of the real radicals of ideals $I = (\mathbf{f})$ and apply this transformation for the computation of S-radicals.

4.2.1 Generic truncated positive linear functionals and real radicals

We are going to use as starting point for our computations annihilators of generic truncated positive linear functionals, or in other words kernels of moment matrices. Indeed, generic elements (see Definition 3.4.9) can be used to compute the real radical of ideals, see [Ros09, th. 7.39]. We give in Theorem 4.2.1 a simpler proof of this result. See also Theorem 3.4.11 for a generalization to quadratic modules.

Theorem 4.2.1. *Let $\Lambda^* \in \mathcal{L}_{2d}(\pm\mathbf{h})$ be generic and $I = (\mathbf{h})$. Then for every $d \geq \deg\mathbf{h}$ we have $I \subset (\operatorname{Ann}_d(\Lambda^*)) \subset \sqrt[\mathbb{R}]{I}$. Moreover for d big enough $(\operatorname{Ann}_d(\Lambda^*)) = \sqrt[\mathbb{R}]{I}$.*

Proof. The inclusion $I \subset (\operatorname{Ann}_d(\Lambda^*))$ is clear since $\mathbf{h} \subset \operatorname{Ann}_d(\Lambda^*)$ by definition. Now let $J = \sqrt[\mathbb{R}]{I}$. Notice that, for $\xi \in \mathbb{R}^n$, $\operatorname{Ann}_d(\mathbf{e}_\xi) = \mathcal{I}(\xi)_d = (x_1 - \xi_1, \dots, x_n - \xi_n)_d$. Moreover, if $\xi \in \mathcal{V}_{\mathbb{R}}(I)$, then $\mathbf{e}_\xi^{[2d]} \in \mathcal{L}_{2d}(\pm\mathbf{h})$. Then, since Λ^* is generic:

$$\operatorname{Ann}_d(\Lambda^*) \subset \bigcap_{\xi \in \mathcal{V}_{\mathbb{R}}(I)} \operatorname{Ann}_d(\mathbf{e}_\xi) = \bigcap_{\xi \in \mathcal{V}_{\mathbb{R}}(I)} \mathcal{I}(\xi)_d = J_d,$$

and thus $(\operatorname{Ann}_d(\Lambda^*)) \subset J$.

For the second part, let $g_1, \dots, g_k$ be generators of J. By the Real Nullstellensatz, $\forall i$ there exists $m_i \in \mathbb{N}, s_i \in \Sigma^2$ such that $g_i^{2^{m_i}} + s_i \in I$. Then for d big enough and $\Lambda \in \mathcal{L}_{2d}(\pm\mathbf{h})$ we have $\langle \Lambda^{[2d]}, g_i^{2^{m_i}} + s_i \rangle = 0$, thus $\langle \Lambda^{[2d]}, g_i^{2^{m_i}} \rangle = 0$ and $g_i \in \operatorname{Ann}_d(\Lambda)$. This implies $J \subset (\operatorname{Ann}_d(\Lambda))$ for all $\Lambda \in \mathcal{L}_{2d}(\pm\mathbf{h})$, and in particular for $\Lambda = \Lambda^*$ generic. □

Notice that in Theorem 3.4.11 we need to discard high degree pseudo-moments, or in other words to restrict the annihilator to some subspace, while in Theorem 4.2.1 it is not necessary, since we have only equations and no inequalities in the description.

The goal of the paper is to find an effective algorithm, based on Theorem 4.2.1, to compute $\sqrt[\mathbb{R}]{I}$. In the case of a finite real variety, the flat extension criterion [LLR08; Las+13] certifies that $(\operatorname{Ann}_d(\Lambda^*)) = \sqrt[\mathbb{R}]{I}$ for some $d \in \mathbb{N}$. We will focus in the positive dimensional case, when such a criterion cannot apply (for instance, as a consequence of Theorem 3.5.4 with f constant and $Q = \mathcal{Q}(\pm\mathbf{f}) = (\mathbf{f}) + \Sigma^2$).

Notice that the inclusion $I \subset (\mathrm{Ann}_d(\Lambda^*)) \subset \sqrt[\mathbb{R}]{I}$ implies $\mathcal{V}_{\mathbb{R}}(\mathrm{Ann}_d(\Lambda^*)) = \mathcal{V}_{\mathbb{R}}(I)$ for all d, i.e. the real zero locus does not change increasing d. On the contrary, the complex variety can change, and there might be also embedded components.

4.2.2 Numerical algebraic geometry

We briefly summarize the main notions of Numerical Algebraic Geometry, mainly following [SW05], that we will need to design our algorithm.

The goal of Numerical Algebraic Geometry is to study algebraic varieties using numerical analysis. In particular, given a complex algebraic variety $X \subset \mathbb{C}^n$ (or $X \subset \mathbb{P}^n$), we want to be able to produce a numerical encoding of X that allow to answer basic questions about X:

- Given $x \in \mathbb{C}^n$, is x in X?
- What is the dimension and the degree of X?
- What are the irreducible components of X?
- What are the dimension and the degree of these irreducible components?

The key theoretical ingredient to find such a numerical encoding are the so-called *Bertini's Theorems*. There are several versions of these theorems, but the general principle behind them may be stated as follows: if a variety X has a certain property, then a sufficiently general affine hyperplane section of X has the same property. This is especially useful in induction arguments. We recall one of these theorems, taken from [SW05, th. 13.2.1]. See also [SW05, th. A.7.1, th. A.8.7 th. A.9.2] for other versions used in numerical algebraic geometry, and [Sha13a; Har77; Jou83] for more.

Theorem 4.2.2 (Bertini's Theorem). *Let $X \subset \mathbb{C}^n$ be an irreducible variety. Then, if $L(\mathbf{a})$ is the affine hyperplane given by $a_0 + a_1x_1 + \cdots + a_nx_n = 0$, then there exists an open dense $U \subset \mathbb{C}^{n+1}$ such that if $\mathbf{a} = (a_0, \ldots, a_n) \in U$, then:*

- *if* $\dim X = 0$, *then* $X \cap L(\mathbf{a}) = \emptyset$;
- *if* $\dim X > 0$, *then* $X \cap L(\mathbf{a})$ *is irreducible of dimension* $\dim X - 1$, *and degree equal to* $\deg X$.

In particular using Theorem 4.2.2 we can easily see that for generic $(\mathbf{a}_1, \ldots \mathbf{a}_{n-\dim X}) \in \mathbb{C}^{(n+1)\times(n-\dim X)}$, the affine space $L = L(\mathbf{a}_1) \cap \cdots \cap L(\mathbf{a}_{n-\dim X})$ is such that $L \cap X = \{w_1, \ldots, w_{\deg X}\}$ is equal to $\deg X$ points. It turns out that these affine space L, the points $\{w_1, \ldots, w_{\deg X}\}$ and equations $\mathbf{f} = f_1, \ldots f_r$ such that $\mathcal{V}_{\mathbb{C}}(\mathbf{f}) \supset X$ are the numerical encoding of varieties that we were looking for.

Definition 4.2.3. Let $\mathbf{f} = f_1, \ldots f_r$ and X be an irreducible component of $\mathcal{V}_{\mathbb{C}}(\mathbf{f})$. Then a *witness set* for X is a triple $(\mathbf{f}, L, W)$, where L is a generic affine space of dimension $n - \dim X$ and $W = \{w_1, \ldots, w_{\deg X}\} = X \cap L$ is the set of *witness points*.

If $\mathcal{V}_{\mathbb{C}}(\mathbf{f}) = X_1 \cup \cdots \cup X_m$ is the irreducible decomposition of $\mathcal{V}_{\mathbb{C}}(\mathbf{f})$, then a *numerical irreducible decomposition* is a collection of witness sets $(\mathbf{f}, L_i, W_i)$ for every irreducible component X_i of $\mathcal{V}_{\mathbb{C}}(\mathbf{f})$ such that all the witness points W_i are disjoint.

Notice that witness sets can be introduced for equidimensional components of a variety (that is, for the union of the components of the same dimension in an irreducible decomposition), see for instance [SVW01; SW05]. Since in the following we will use witness sets only for irreducible varieties, we defined witness sets only in our case of interest.

The numerical irreducible decomposition of $\mathcal{V}_{\mathbb{C}}(\mathbf{f})$ as a collection of witness sets provides a description of all the irreducible components X_i associated to the isolated primary components Q_i of $I = (\mathbf{f})$ [AM94]. To check that these primary components are reduced and thus prime (i.e. $\sqrt{Q_i} = Q_i$), it is enough to check that the Jacobian of $\mathbf{f}$ is of rank $n - \dim X_i$ (Jacobian criterion) at one of the sample points of the witness set W_i, describing the irreducible component $X_i = \mathcal{V}_{\mathbb{C}}(Q_i)$. Several methods, based on homotopy techniques, have been developed over the past to compute the numerical irreducible decomposition, see e.g. [SVW01; HSW11; Bat+13].

Checking that $I = (\mathbf{f})$ has no embedded component can also be done by numerical irreducible decomposition of deflated ideals, as described in [KL17]. We are not going to use this deflation technique to check non-embedded components.

A numerical irreducible decomposition of an algebraic variety X can also be used to compute defining equations $\mathbf{h}_i = h_{i,1}, \dots, h_{i,n+1}$ such that $\mathcal{V}_{\mathbb{C}}(\mathbf{h}_i) = X_i$ for every irreducible component X_i. In particular, homotopy techniques are employed to generate enough sample points on X_i. The equations $h_{i,j}$ are then computed by projection of the sample points onto $\leq n+1$ generic linear spaces of dimension $(\dim(X)+1)$ and by interpolation. See e.g. [SVW01], for more details, and Lemma 4.4.6.

All the construction introduced above are efficiently implemented in `Bertini`[Bat+]. Another software for numerical algebraic geometry is `HomotopyContinuation.jl` [BT18].

4.3 Orthogonal polynomials and annihilator

To compute the real radical, we need to compute a basis of the annihilator of a truncated positive linear functional $\Lambda \in (\mathbb{R}[\mathbf{x}]_{2d})^*$ such that $\langle \Lambda, p^2 \rangle \geq 0$ for $p \in \mathbb{R}[\mathbf{x}]_d$ (or, equivalently, such that $H_\Lambda^d \succcurlyeq 0$). In this section, we describe an efficient algorithm to compute a basis of $\mathrm{Ann}_d(\Lambda)$. Recall that, for $p \in \mathbb{R}[\mathbf{x}]_d$ and $\Lambda \in \mathbb{R}[\mathbf{x}]_{2d}^*$ such that $H_\Lambda^d \succcurlyeq 0$, $\left\langle \Lambda \middle| h^2 \right\rangle = 0$ if and only if $(h \star \Lambda)^{[d]} = 0$. Therefore:

$$\mathrm{Ann}_d(\Lambda) = \{p \in \mathbb{R}[\mathbf{x}]_d \mid p \star \Lambda = 0\} = \{p \in \mathbb{R}[\mathbf{x}]_d \mid \left\langle \Lambda \middle| p^2 \right\rangle = 0\}.$$

Our algorithm to compute a basis of $\mathrm{Ann}_d(\Lambda)$ is a Gram-Schmidt orthogonalization process, using the inner product $\langle \cdot, \cdot \rangle_\Lambda$ defined, for $p, q \in \mathbb{R}[\mathbf{x}]_d$, by

$$\langle p, q \rangle_\Lambda := \langle \Lambda | p\, q \rangle.$$

By ordering the monomials basis of $\mathbb{R}[\mathbf{x}]_d$ and projecting successively a monomial $\mathbf{x}^\alpha$ onto the space spanned by the previous monomials, we construct monomial basis $\mathbf{b} = \{\mathbf{x}^\beta\}$ of $\mathbb{R}[\mathbf{x}]_d / \mathrm{Ann}_d(\Lambda)$, a corresponding basis of orthogonal polynomials $\mathbf{p} = (p_\beta)$ and a basis $\mathbf{k} = (k_\gamma)$ of $\mathrm{Ann}_d(\Lambda)$. The orthogonal polynomials are such that

$$\langle p_\beta, p_{\beta'} \rangle_\Lambda = \begin{cases} > 0 & \text{if } \beta = \beta' \\ 0 & \text{otherwise,} \end{cases}$$

and for all β,γ, we have $\langle p_\beta, k_\gamma\rangle_\Lambda = \langle k_\gamma, k_\gamma\rangle_\Lambda = \left\langle \Lambda \middle| k_\gamma^2 \right\rangle = 0$.

To compute these polynomials, we use a projection defined on the orthogonal of the space spanned by orthogonal polynomials $\mathbf{p} = p_1,\dots,p_l$ such that $\langle p_i, p_i\rangle_\Lambda > 0$ and $\langle p_i, p_j\rangle_\Lambda = 0$ if $i \neq j$, as follows: for $f \in \mathbb{R}[\mathbf{x}]_d$,

$$\operatorname{proj}(f,\mathbf{p}) = f - \sum_{i=1}^{l} \frac{\langle f, p_i\rangle_\Lambda}{\langle p_i, p_i\rangle_\Lambda} p_i.$$

By construction, we have:

$$\langle \operatorname{proj}(f,\mathbf{p}), p_j\rangle_\Lambda = \langle f - \sum_{i=1}^{l} \frac{\langle f, p_i\rangle_\Lambda}{\langle p_i, p_i\rangle_\Lambda} p_i, p_j\rangle_\Lambda = \langle f, p_j\rangle_\Lambda - \sum_{i=1}^{l} \frac{\langle f, p_i\rangle_\Lambda}{\langle p_i, p_i\rangle_\Lambda} \langle p_i, p_j\rangle_\Lambda = 0$$

for $j = 1,\dots,l$. In practice, the implementation of this projection is done by the so-called Modified Gram-Schmidt projection algorithm, which is known to have a better numerical behavior than the direct Gram-Schmidt orthogonalization process [TB97, Lecture 8].

To compute a basis of $\operatorname{Ann}_d(\Lambda)$, we choose a monomial ordering $\prec$ compatible with the degree (e.g. the graded reverse lexicographic ordering), see Section 1.1.3. We build the list of monomials $\mathbf{s}$ of degree $\leq d$ in increasing order for this total ordering $\prec$. Algorithm 4.3.1 chooses incrementally a new monomial in the list $\mathbf{s}$ and projects it on the space spanned by the previous orthogonal polynomials. The new monomials computed by the function $\operatorname{next}(\mathbf{s},\mathbf{b},\mathbf{l})$ are the monomials with the lowest degree in $\mathbf{s}$, ordered w.r.t. $\prec$, not in $\mathbf{b}$ and not divisible by a monomial of $\mathbf{l}$:

$$\operatorname{next}(\mathbf{s},\mathbf{b},\mathbf{l}) := \{\mathbf{x}^\alpha \in \mathbf{s} \mid |\alpha| \text{ is minimal, } \mathbf{x}^\alpha \notin \mathbf{b},\ \alpha \ngeq \gamma\ \forall \mathbf{x}^\gamma \in \mathbf{l}\},$$

where $\alpha = (\alpha_1,\dots,\alpha_n) \ngeq \gamma = (\gamma_1,\dots,\gamma_n)$ if there exists j such that $\alpha_j < \gamma_j$. We are now ready to describe the algorithm.

By construction, the vector space spanned by $\mathbf{b}$ and $\mathbf{p}$ are equal at each loop of the algorithm. As the function $\operatorname{next}(\mathbf{s},\mathbf{b},\mathbf{l})$ outputs monomials in $\mathbf{s}$ greater than $\mathbf{b}$ then the monomials in $\mathbf{n}$ are greater than the monomials in $\mathbf{b}$. Thus, the leading term of $k_\gamma \in \mathbf{k}$ is $\mathbf{x}^\gamma$.

Let $\mathbf{k}$, $\mathbf{l}$, $\mathbf{p}$, $\mathbf{b}$ denote the output of Algorithm 4.3.1. For $\alpha \in \mathbb{N}^n$, let $(\mathbf{k})_{\preceq\alpha}$ be the vector space spanned by the elements of the form $\mathbf{x}^\delta k_\gamma$ with $\delta + \gamma \preceq \alpha$. Similarly, $\mathbf{p}_{\preceq\alpha}$ is the set of $p_\beta \in \mathbf{p}$ such that $\beta \preceq \alpha$. We prove that $\mathbf{k}$ is a Grobner basis of $\operatorname{Ann}_d(\Lambda)$, that is any element of $\operatorname{Ann}_d(\Lambda)$ reduces to 0 by $\mathbf{k}$:

Proposition 4.3.1. *Let $\Lambda \in \mathbb{R}[\mathbf{x}]^*_{2d+2}$, $\mathbf{k},\mathbf{p}$ be the output of Algorithm 4.3.1. For $\mathbf{x}^\alpha \in (\mathbf{l})_d$, i.e. divisible by a monomial in $\mathbf{l}$ and of degree $|\alpha| \leq d$, $p_\alpha = \operatorname{proj}(\mathbf{x}^\alpha, \mathbf{p}_{\prec\alpha})$ is in $(\mathbf{k})_{\preceq\alpha} \subset \operatorname{Ann}_d(\Lambda)$.*

Proof. Let us prove it by induction on the ordering of α. The lowest element in $(\mathbf{l})_d$ is a monomial $\mathbf{x}^\gamma$ of $\mathbf{l}$. As $k_\gamma = \operatorname{proj}(\mathbf{x}^\gamma, \mathbf{p}_{\preceq\gamma})$ is such that $\langle k_\gamma, k_\gamma\rangle_\Lambda = \left\langle \Lambda \middle| k_\gamma^2\right\rangle = 0$, $k_\gamma = \operatorname{proj}(\mathbf{x}^\gamma, \mathbf{p}_{\preceq\gamma}) \in (\mathbf{k})_{\preceq\gamma} \subset \operatorname{Ann}_d(\Lambda)$. Then the induction hypothesis is true for the lowest monomial of $(\mathbf{l})_d$.

Assume that it is true for $\mathbf{x}^{\alpha'} \in (\mathbf{l})_d$ and for all the smaller monomials w.r.t. $\prec$. Let $\mathbf{x}^\alpha$ be the next monomial in $(\mathbf{l})_d$ for the monomial ordering $\prec$. Then, there exists $\mathbf{x}^{\alpha''} \in (\mathbf{l})_{\preceq\alpha'}$ and $i_0 \in 1,\dots,n$ such that $x_{i_0}\mathbf{x}^{\alpha''} = \mathbf{x}^\alpha$. As $p_\alpha - x_{i_0} p_{\alpha''}$ has a leading term smaller that $\mathbf{x}^\alpha$, it can be written as a linear combination of $p_{\alpha'} = \operatorname{proj}(\mathbf{x}^{\alpha'}, \mathbf{p}_{\prec\alpha'})$ with $\alpha' \prec \alpha$. More precisely, we have

$$p_\alpha = x_{i_0} p_{\alpha''} + \sum_{\delta \prec \alpha, \mathbf{x}^\delta \in (\mathbf{l})_d} \lambda_\delta p_\delta + \sum_{\beta \prec \alpha, \mathbf{x}^\beta \in \mathbf{b}} \mu_\beta p_\beta,$$

Algorithm 4.3.1: Orthogonal polynomials and annihilator of Λ

Input:A linear functional $\Lambda \in \mathbb{R}[\mathbf{x}]^*_{2d+2}$ such that $H_\Lambda^{d+1} \succcurlyeq 0$.

- Let $\mathbf{b} := []$; $\mathbf{p} := []$; $\mathbf{k} := []$; $\mathbf{l} = []$; $\mathbf{n} := [1]$; $\mathbf{s} := [\mathbf{x}^\alpha, |\alpha| \le d]$;
- while $\mathbf{n} \neq \emptyset$ do
  - for each $\mathbf{x}^\alpha \in \mathbf{n}$,
    (i) $p_\alpha := \mathrm{proj}(\mathbf{x}^\alpha, \mathbf{p})$;
    (ii) compute $v_\alpha = \langle p_\alpha, p_\alpha \rangle_\Lambda$;
    (iii) if $v_\alpha \neq 0$ then
      add $\mathbf{x}^\alpha$ to $\mathbf{b}$; add p_α to $\mathbf{p}$;
    else
      add $k_\alpha := p_\alpha$ to $\mathbf{k}$; add $\mathbf{x}^\alpha$ to $\mathbf{l}$;
    end;
  - $\mathbf{n} := \mathrm{next}(\mathbf{s}, \mathbf{b}, \mathbf{l})$;

Output:

- a basis $\mathbf{k} = [k_\gamma]_{\mathbf{x}^\gamma \in \mathbf{l}}$ of $\mathrm{Ann}_d(\Lambda)$ and their leading monomials $\mathbf{l} = [\mathbf{x}^\gamma]$;
- a basis of orthogonal polynomials $\mathbf{p} = [p_\beta]$;
- a monomial set $\mathbf{b} = [\mathbf{x}^\beta]$.

for some $\lambda_\delta, \mu_\beta \in \mathbb{R}$.

By induction hypothesis, $p_{\alpha''}, p_\delta \in (\mathbf{k})_{\le\alpha'} \subset (\mathbf{k})_{\le\alpha} \subset \mathrm{Ann}_d(\Lambda)$. Moreover, as $p_{\alpha''} \in \mathrm{Ann}_d(\Lambda) \subset \mathrm{Ann}_{d+1}(\Lambda)$, for any $p \in \mathbb{R}[\mathbf{x}]_d$ we have $\langle x_{i_0} p_{\alpha''}, p \rangle_\Lambda = \langle p_{\alpha''}, x_{i_0} p \rangle_\Lambda = 0$. This shows that $x_{i_0} p_{\alpha''} \in (\mathbf{k})_{\le\alpha} \cap \mathrm{Ann}_d(\Lambda)$.

By definition of $p_\alpha = \mathrm{proj}(\mathbf{x}^\alpha, \mathbf{p}_{\prec\alpha})$, $\langle p_\alpha, p_\beta \rangle_\Lambda = 0$ for $\mathbf{x}^\beta \in \mathbf{b}_{\prec\alpha}$ so that $\mu_\beta = \frac{\langle p_\alpha, p_\beta \rangle_\Lambda}{\langle p_\beta, p_\beta \rangle_\Lambda} = 0$ and $p_\alpha \in (\mathbf{k})_{\le\alpha} \cap \mathrm{Ann}_d(\Lambda)$.

As $(\mathbf{k})_{\le\alpha} = (\mathbf{k})_{\le\alpha'} + \langle p_\alpha \rangle$, we have $(\mathbf{k})_{\le\alpha} \subset \mathrm{Ann}_d(\Lambda)$, which proves the induction hypothesis for α and concludes the proof. □

This proposition explains why the function $\mathrm{next}(\mathbf{s}, \mathbf{b}, \mathbf{l})$ only outputs the monomials with the lowest degree in $\mathbf{s}$, ordered w.r.t. $\prec$, not in $\mathbf{b}$ and not divisible by a monomial of $\mathbf{l}$.

This algorithm is an optimization of Algorithm 4.1 in [Mou18] or Algorithm 3.2 in [Mou17]. It strongly exploits the positivity of the linear functional Λ and improves significantly the performance. We will present its behavior in Section 4.6 in real instances, where Λ is given as an approximate sequence of pseudo-moments, and hereafter in some easy, exact cases to illustrate the algorithm.

Example 4.3.2. Let $\Lambda \in \mathbb{R}[\mathbf{x}]^*_{2d+2}$ be such that $H_\Lambda^{d+1} \succcurlyeq 0$ and $\langle \Lambda | 1 \rangle = \langle 1, 1 \rangle_\Lambda = 0$. In the first iteration of Algorithm 4.3.1, we have $\mathbf{l} = [1]$. Therefore $\mathrm{next}(\mathbf{s}, \mathbf{b}, \mathbf{l}) = \emptyset$, and the algorithm stops. This is coherent with Lemma 1.3.9, as in this case we have $\Lambda^{[d+1]} = 0$.

Example 4.3.3. Let $\Lambda = \mathbf{e}_{(0,0)}^{[2d+2]} + \mathbf{e}_{(0,1)}^{[2d+2]} \in \mathbb{R}[x,y]_{2d+2}^*$, $d \geq 1$. Since Λ is induced by a measure, $H_\Lambda^{d+1} \succcurlyeq 0$. We follow the steps of Algorithm 4.3.1.

We start with $\mathbf{b} := []$; $\mathbf{p} := []$; $\mathbf{k} := []$; $\mathbf{l} = []$; $\mathbf{n} := [1]$; $\mathbf{s} := [\mathbf{x}^\alpha, |\alpha| \leq d]$. The only monomial in $\mathbf{b}$ is $\mathbf{x}^\alpha = 1$, $\alpha = (0,0)$. Then $p_{(0,0)} = 1$ and $v_{(0,0)} = 2$. Therefore we set $\mathbf{b} = [1]$ and $\mathbf{p} = [1]$ and move on.

On the second loop, we have $\mathbf{n} = \text{next}(\mathbf{s},[1],[]) = [x,y]$. We start with $\mathbf{x}^\alpha = x$, $\alpha = (1,0)$. Then:

$$p_{(1,0)} = \text{proj}(x,\mathbf{p}) = x - \frac{\langle x,1\rangle_\Lambda}{\langle 1,1\rangle\Lambda} 1 = x - \frac{1}{2}$$
$$v_{(1,0)} = \langle p_{(1,0)}, p_{(1,0)}\rangle_\Lambda = \left\langle \Lambda \middle| p_{(1,0)}^2 \right\rangle = \left\langle \mathbf{e}_{(0,0)} \middle| p_{(1,0)}^2 \right\rangle + \left\langle \mathbf{e}_{(1,0)} \middle| p_{(1,0)}^2 \right\rangle = \frac{1}{2}$$

Therefore we set $\mathbf{b} = [1,x]$ and $\mathbf{p} = [1, x - \frac{1}{2}]$, and move on. We have now $\mathbf{x}^\alpha = y$, $\alpha = (0,1)$. Then:

$$p_{(0,1)} = \text{proj}(y,[1,x-\frac{1}{2}]) = y - \frac{\langle y,1\rangle_\Lambda}{\langle 1,1\rangle_\Lambda} 1 - \frac{\langle y, x-\frac{1}{2}\rangle_\Lambda}{\langle x-\frac{1}{2}, x-\frac{1}{2}\rangle\Lambda}(x-\frac{1}{2}) = y$$
$$v_{(0,1)} = \langle p_{(0,1)}, p_{(0,1)}\rangle_\Lambda = \left\langle \Lambda \middle| p_{(0,1)}^2 \right\rangle = \left\langle \mathbf{e}_{(0,0)} \middle| p_{(0,1)}^2 \right\rangle + \left\langle \mathbf{e}_{(1,0)} \middle| p_{(0,1)}^2 \right\rangle = 0$$

We set then $\mathbf{k} = [y]$ and $\mathbf{l} = [y]$.

On the third loop, we have $\mathbf{n} = \text{next}(\mathbf{s},[1,x],[y]) = [x^2]$. Notice that we do not have to consider xy and y^2 since they are multiples of y. Then we have $\mathbf{x}^\alpha = x^2$, $\alpha = (2,0)$, and:

$$p_{(2,0)} = \text{proj}(x^2,[1,x-\frac{1}{2}]) = x^2 - \frac{\langle x^2,1\rangle_\Lambda}{\langle 1,1\rangle_\Lambda} 1 - \frac{\langle x^2, x-\frac{1}{2}\rangle_\Lambda}{\langle x-\frac{1}{2}, x-\frac{1}{2}\rangle\Lambda}(x-\frac{1}{2}) = x^2 - x$$
$$v_{(2,0)} = \langle p_{(2,0)}, p_{(2,0)}\rangle_\Lambda = \left\langle \Lambda \middle| p_{(2,0)}^2 \right\rangle = \left\langle \mathbf{e}_{(0,0)} \middle| p_{(2,0)}^2 \right\rangle + \left\langle \mathbf{e}_{(1,0)} \middle| p_{(2,0)}^2 \right\rangle = 0$$

Then we set $\mathbf{k} = [y, x^2 - x]$ and $\mathbf{l} = [y, x^2]$.

Since any monomials of degree ≥ 3 is divisible by either y or by x^2, we obtain by definition $\mathbf{n} = \text{next}(\mathbf{s},[1,x],[y,x^2]) = \emptyset$ and the algorithms stops.

Finally, notice that the result is correct, since:

$$\text{Ann}_d(\Lambda) = \text{Ann}_d(\mathbf{e}_{(0,0)}) \cap \text{Ann}_d(\mathbf{e}_{(1,0)}) = (x,y)_d \cap (x-1,y)_d = (x^2-x,y)_d$$

Remark. When the real variety $\mathcal{V}_\mathbb{R}(\mathbf{f})$ is finite, the flat extension test on the rank of H_Λ^k can be replaced by testing that the set $\mathbf{l}$ of initial terms contains a power of each variable x_i, as in Example 4.3.3. This is equivalent to the fact that $\mathbb{R}[\mathbf{x}]/(\mathbf{k})$ is finite dimensional or equivalently that the rank of H_Λ^d is constant for $d \gg 0$.

4.4 Real irreducible components

We introduce an effective algorithm for testing real radicality in the irreducible case.

4.4.1 Genericity

Let $\mathbb{C}^N$ be the N-dimensional affine space and $\mathbb{C}[t_1,\ldots,t_N] = \mathbb{C}[\mathbf{t}]$ be its coordinate (polynomial) ring. We say that a property holds *generically* in $\mathbb{C}^N$ if there exists finitely many nonzero polynomials $\phi_1,\ldots,\phi_l \in \mathbb{C}[\mathbf{t}]$ such that, for $\xi \in \mathbb{C}^N$, when $\phi_1(\xi) \neq 0,\ldots,\phi_l(\xi) \neq 0$ the property holds for ξ.

In particular we will consider linear maps $A \in \hom_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^{k+1})$ as elements in $\mathbb{C}^{n(k+1)}$ in the natural way, and thus talk about *generic linear maps*.

4.4.2 Smooth complex and real zeros

We are interested in tests to determine if a given ideal is real or not. To do so, we introduce the definition of smooth zero of an ideal, in particular in the real case (see [Mar08]). We refer to [Sha13a] for the complex case.

Definition 4.4.1. Let $I = (f_1,\ldots,f_m) \subset \mathbb{R}[\mathbf{x}]$ be a prime ideal and $X = \mathcal{V}_{\mathbb{C}}(I)$. We say that $\xi \in \mathcal{V}_{\mathbb{R}}(I)$ is a *smooth zero* of I if $\operatorname{rank}\operatorname{Jac}(f_1,\ldots,f_m)(\xi) = n - \dim X$.

In other words, a point $\xi \in \mathcal{V}_{\mathbb{R}}(I)$ is a smooth zero of I if and only if ξ is a smooth point of the complex variety $X = \mathcal{V}_{\mathbb{C}}(I)$.

From the Nullstellensatz we deduce that the mapping $V \mapsto \mathcal{I}_{\mathbb{C}}(V)$ is a bijection between irreducible varieties in $\mathbb{C}^n$ and prime ideals in $\mathbb{C}[\mathbf{x}]$. Moreover, for a prime ideal $\mathfrak{p}$, smooth points of $\mathcal{V}_{\mathbb{C}}(\mathfrak{p})$ are dense. On the other hand, from the Real Nullstellensatz we deduce that the mapping $X \mapsto \mathcal{I}_{\mathbb{R}}(X)$ is a bijection between irreducible varieties in $\mathbb{R}^n$ and real prime ideals in $\mathbb{R}[\mathbf{x}]$. For prime ideals $\mathfrak{p} \subset \mathbb{R}[\mathbf{x}]$ which are not real radical, smooth zeros of $\mathfrak{p}$ are not dense in $\mathcal{V}_{\mathbb{R}}(\mathfrak{p})$.

Example 4.4.2. Here are examples of reducible and irreducible algebraic varieties with dense complex smooth points but with no real smooth point.

- $\mathfrak{p} = (x^2 + y^2) \subset \mathbb{R}[x,y]$ is a prime, non real radical ideal, as $\mathcal{V}_{\mathbb{R}}(\mathfrak{p}) = \{(0,0)\}$ and $\sqrt[\mathbb{R}]{\mathfrak{p}} = (x,y)$. $\mathfrak{p}$ does not have smooth real zeros. Notice that $(x^2 + y^2) \subset \mathbb{C}[x,y]$ is not prime, since $x^2 + y^2 = (x+iy)(x-iy)$.

- $\mathfrak{p} = (x^2 + y^2 + z^2) \subset \mathbb{R}[x,y,z]$ is a prime, non real radical ideal, as $\mathcal{V}_{\mathbb{R}}(\mathfrak{p}) = \{(0,0,0)\}$ and $\sqrt[\mathbb{R}]{\mathfrak{p}} = (x,y,z)$. $\mathfrak{p}$ does not have smooth real zeros. In this case $(x^2 + y^2 + z^2) \subset \mathbb{C}[x,y,z]$ is prime, since $x^2 + y^2 + z^2$ is irreducible over $\mathbb{C}$.

We recall criterions for testing whether a prime ideal $\mathfrak{p} \subset \mathbb{R}[\mathbf{x}]$ is real radical or not, based on the detection of smooth zeros.

Theorem 4.4.3 (Simple Point Criterion [Mar08, th. 12.6.1])**.** *Let $\mathfrak{p}$ be a prime ideal of $\mathbb{R}[\mathbf{x}]$. The following are equivalent:*

- $\mathfrak{p}$ *is a real ideal;*

- $\mathfrak{p} = \mathcal{I}_{\mathbb{R}}(\mathcal{V}_{\mathbb{R}}(\mathfrak{p}))$*;*

- $\operatorname{cl}(\mathcal{V}_{\mathbb{R}}(\mathfrak{p})) = \mathcal{V}_{\mathbb{C}}(\mathfrak{p})$*;*

- $\mathfrak{p}$ *has a smooth real zero.*

We say that $X \subset \mathbb{C}^n$ is *defined over* $\mathbb{R}$ if the vanishing ideal $\mathcal{I}_{\mathbb{C}}(X)$ is generated by real polynomials: formally, if there exist $f_1, \ldots f_r \in \mathbb{R}[\mathbf{x}] \subset \mathbb{C}[\mathbf{x}]$ such that $\mathcal{I}_{\mathbb{C}}(X) = (f_1, \ldots f_r)$. Equivalently, X is defined over $\mathbb{R}$ if and only if X is invariant by complex conjugation: $X = \operatorname{conj}(X)$, where $\operatorname{conj}: \mathbb{C}^n \to \mathbb{C}^n$, $(x_1 + iy_1, \ldots, x_n + iy_n) \mapsto (x_1 - iy_1, \ldots, x_n - iy_n)$.

Let $X \subset \mathbb{C}^n$ be an irreducible variety defined over $\mathbb{R}$ and $I = \mathcal{I}_{\mathbb{C}}(X) \cap \mathbb{R}[\mathbf{x}] \subset \mathbb{R}[\mathbf{x}]$ the ideal defined by its real generators. If follows from Theorem 4.4.3 that $X_{\mathbb{R}} = \mathcal{V}_{\mathbb{R}}(I)$ is Zariski dense in X if and only if I is a real prime ideal. In this case we say that X is *totally real*.

For hypersurfaces there exists another criterion based on the change of sign of the defining polynomial.

Theorem 4.4.4 (Sign Changing Criterion [Mar08, th. 12.7.1])**.** *Let $f \in \mathbb{R}[\mathbf{x}]$ be an irreducible polynomial. The following are equivalent:*

- *(f) is a real ideal;*
- *(f) has a smooth real point (i.e. there exists $\xi \in \mathcal{V}_{\mathbb{R}}(I)$ such that $\nabla f(\xi) \neq 0$);*
- *the polynomial f changes sign in $\mathbb{R}^n$ (i.e. there exists $x, y \in \mathbb{R}^n$ such that $f(x)f(y) < 0$).*

4.4.3 Test for real radicality

We reduce the problem of testing real radicality to the hypersurface case, and then use the Simple Point Criterion. For that purpose we project $X \subset \mathbb{C}^n$, irreducible variety of dimension k, on a linear subspace $\mathbb{C}^{k+1} \subset \mathbb{C}^n$, in such a way X and $\operatorname{cl}(\pi(X))$ are *birational*. (see Section 1.1.4 or [Sha13a, p. 38] for the definition).

It is classical that every irreducible (complex, affine) variety is birational to an hypersurface. We recall briefly this result to show that we can choose a generic projection as birational morphism, as done for the geometric resolution or rational representation, see for instance [Lec03] or [Bos+17].

Lemma 4.4.5. *Let $X \subset \mathbb{C}^n$ be an irreducible varierty of dimension k and $\pi: \mathbb{C}^n \to \mathbb{C}^{k+1}$ be a generic projection. Then X is birational to $\pi(X)$, i.e. $X \cong \operatorname{cl}(\pi(X))$.*

Proof. (sketch) We show that the birational morphism in [Sha13b, p. 39] can be given as a generic projection. Indeed, $\mathbb{C}(X)$ if a finite field extension of $\mathbb{C}$ (we can see this for instance choosing the monomials $x_1, \ldots, x_n$) with trascendence degree k. We can choose algebraically independent elements $\ell_1, \ldots, \ell_k$, generic linear forms in the indeterminates $\mathbf{x}$, such that $\mathbb{C}[\ell_1, \ldots, \ell_k]$ is a subring of $\mathbb{C}[X]$, and the extension $\mathbb{C}(\ell_1, \ldots \ell_k) \subset \mathbb{C}(X)$ is finite extension (see for instance [Eis04, cor. 16.18]). From the primitive element theorem (see for instance [Art17, th. 15.8.1]) one can choose a primitive element l_{k+1} for the extension as a generic linear form. Then $\ell_1, \ldots, \ell_{k+1}$ define the (generic) projection $\pi: \mathbb{C}^n \to \mathbb{C}^{k+1}$, $\xi \mapsto (\ell_1(\xi), \ldots, \ell_{k+1}(\xi))$ and X is birational to $\operatorname{cl}(\pi(X))$. □

Another invariant via generic projection is the degree. This result can be used to compute equations for an (irreducible) algebraic variety from enough generic projections.

Lemma 4.4.6 ([SVW01, sec. 5.2])**.** *Let $X \subset \mathbb{C}^n$ be an irreducible varierty of dimension k and $\pi: \mathbb{C}^n \to \mathbb{C}^{k+1}$ be a generic projection. Then* $\deg X = \deg \operatorname{cl}(\pi(X))$*. Furthermore, if $\pi_1, \ldots, \pi_{n+1}$ are generic projections, then, if h_i is the equation of the hypersurface* $\operatorname{cl}(\pi_i(X))$*, we have:*

$$\mathcal{V}_{\mathbb{C}}(h_0(\pi_0(\mathbf{x})), \ldots, h_{n+1}(\pi_{n+1}(\mathbf{x}))) = X.$$

Moreover, the equations define a radical ideal: $(h_0(\pi_0(\mathbf{x})),\ldots,h_{n+1}(\pi_{n+1}(\mathbf{x}))) = \mathcal{V}_{\mathbb{C}}(X)$.

We choose a generic projection defined over $\mathbb{R}$. In this case we show that X has a smooth real point if and only if $\mathrm{cl}(\pi(X))$ has a smooth real point, using the following propositions.

Proposition 4.4.7. *Let* $X \subset \mathbb{C}^n$ *be an irreducible varierty defined over* $\mathbb{R}$ *of dimension* k, *and let* $\pi\colon \mathbb{C}^n \to \mathbb{C}^{k+1}$ *be a generic projection defined over* $\mathbb{R}$. *Then* $\mathrm{cl}(\pi(X))$ *is defined over* $\mathbb{R}$ *and if* X *has a smooth real point then* $\mathrm{cl}(\pi(X))$ *has a smooth real point.*

Proof. Let $\pi\colon \mathbb{C}^n \to \mathbb{C}^{k+1}$ be a generic projection defined over $\mathbb{R}$. As X is defined over $\mathbb{R}$, $\mathrm{cl}(\pi(X))$ is also defined over $\mathbb{R}$ since $\mathcal{I}(\pi(X))$ is the elimination ideal $(\mathcal{I}(X) + (\pi(\mathbf{x}) - \mathbf{y})) \cap \mathbb{R}[\mathbf{y}]$, where $\mathbf{y} = y_1,\ldots,y_{k+1}$ are coordinates of $\mathbb{C}^{k+1}$ (see [CLO15]).

If X has a smooth real point then $X_{\mathbb{R}}$ is Zariski dense in X by Theorem 4.4.3. Then $\pi(X_{\mathbb{R}})$ is Zariski dense in $\pi(X)$. Since π is defined over $\mathbb{R}$ we have that $\pi(X_{\mathbb{R}}) \subset (\pi(X))_{\mathbb{R}}$ and $(\pi(X))_{\mathbb{R}}$ is Zariski dense in $\pi(X)$. Then $\mathrm{cl}((\pi(X))_{\mathbb{R}}) = \mathrm{cl}(\pi(X))$ and by Theorem 4.4.3 $\mathrm{cl}(\pi(X))$ has a smooth real point. □

Proposition 4.4.8. *Let* $X \subset \mathbb{C}^n$ *be an irreducible variety defined over* $\mathbb{R}$ *of dimension* k *without smooth real points. Then, for a generic projection* $\pi\colon \mathbb{C}^n \to \mathbb{C}^{k+1}$ *defined over* $\mathbb{R}$, $\mathrm{cl}(\pi(X))$ *is defined over* $\mathbb{R}$ *and has no smooth real points.*

Proof. By Proposition 4.4.7, $\mathrm{cl}(\pi(X))$ is defined over $\mathbb{R}$.

Assume now that $\mathrm{cl}(\pi(X))$ has a smooth real point. Since X is generically birational to $\pi(X)$ (Lemma 4.4.5), the preimage of a generic smooth point in $\pi(X)$ is a single point in X, which is smooth. If π is defined over $\mathbb{R}$ then this smooth point $p \in X$ is real since $\pi(p) = \overline{\pi(p)} = \pi(\overline{p})$ implies that $p = \overline{p}$, showing that X has a smooth real point. □

Proposition 4.4.9. *Let* $X \subset \mathbb{C}^n$ *be an irreducible variety not defined over* $\mathbb{R}$ *of dimension* k. *If* $\pi\colon \mathbb{C}^n \to \mathbb{C}^{k+1}$ *is a generic projection defined over* $\mathbb{R}$ *then* $\mathrm{cl}(\pi(X))$ *is not defined over* $\mathbb{R}$.

Proof. Let $X \subset \mathbb{C}^n$ be an irreducible variety not defined over $\mathbb{R}$ of dimension k. Assume by contradiction that $\mathrm{cl}(\pi(X))$ is defined over $\mathbb{R}$ if $\pi\colon \mathbb{C}^n \to \mathbb{C}^{k+1}$ is a generic projection defined over $\mathbb{R}$. Then let $\pi_1,\ldots,\pi_{n+1}$ be generic projections defined over $\mathbb{R}$: we have $\mathcal{V}_{\mathbb{C}}(h_i) = \mathrm{cl}(\pi(X))$ for $i = 1,\ldots,n+1$, with h_i real polynomials since $\mathrm{cl}(\pi_i(X))$ is defined over $\mathbb{R}$. Therefore, $\mathcal{V}_{\mathbb{C}}(h_0(\pi_0(\mathbf{x})),\ldots,h_{n+1}(\pi_{n+1}(\mathbf{x}))) = X$ from Lemma 4.4.6, where $h_i(\pi_i(\mathbf{x}))$ are real equations since h_i and π_i are real. Thus X is defined over $\mathbb{R}$, a contradiction. This shows that $\mathrm{cl}(\pi(X))$ is not defined over $\mathbb{R}$ if X is not defined over $\mathbb{R}$ and π is a generic real projection. □

Theorem 4.4.10. *Let* $X \subset \mathbb{C}^n$ *be an irreducible variety of dimension* k. *Then* X *is defined over* $\mathbb{R}$ *and has a smooth real point if and only if, for* $\pi\colon \mathbb{C}^n \to \mathbb{C}^{k+1}$ *generic projection defined over* $\mathbb{R}$, $\mathrm{cl}(\pi(X))$ *is defined over* $\mathbb{R}$ *and has a smooth real point.*

Proof. If X has a smooth real point then we apply Proposition 4.4.7 to conclude that $\mathrm{cl}(\pi(X))$ has a smooth real point. If X is defined over $\mathbb{R}$ but has no smooth real point, we apply Proposition 4.4.8 and deduce that $\mathrm{cl}(\pi(X))$ has no smooth real points. Finally, if X is not defined over $\mathbb{R}$ we apply Proposition 4.4.9 to show that $\mathrm{cl}(\pi(X))$ is not defined over $\mathbb{R}$. □

Corollary 4.4.11. *Let* $X \subset \mathbb{C}^n$ *be an irreducible variety of dimension* k, *and* $\pi\colon \mathbb{C}^n \to \mathbb{C}^{k+1}$ *a generic projection defined over* $\mathbb{R}$. *Then the following are equivalent:*

(i) X *is defined over* $\mathbb{R}$ *and the real generators of* $\mathcal{I}(X)$ *define a real radical ideal in* $\mathbb{R}[\mathbf{x}]$;

(ii) $\mathcal{I}(\pi(X))$ *is generated by a real polynomial, irreducible over* $\mathbb{C}$*, which changes sign in* $\mathbb{R}^{k+1}$.

Proof. By Theorem 4.4.3, real generators of $\mathcal{I}(X)$ define a real radical ideal if and only if X has a smooth real point . Then $(i) \iff (ii)$ follows from Theorem 4.4.10 and Theorem 4.4.4. □

We finally describe the algorithm for testing real radicality.

Algorithm 4.4.1: Test real radicality

Input: An irreducible variety $X \subset \mathbb{C}^n$ of dim. k and $\varepsilon, r > 0$.

(i) Fix a generic projection $\pi \colon \mathbb{C}^n \to \mathbb{C}^{k+1}$;

(ii) Compute the irreducible polynomial h defining $\mathrm{cl}(\pi(X))$;

(iii) If h is not a real polynomial return false;

(iv) Choose a generic point $\xi \in \mathbb{R}^{k+1}$ such that $h(\xi) \neq 0$;

(v) $s := \mathrm{sign}(h(\xi))$;

(vi) Let $f = \|\mathbf{x} - \xi\|_2^2$. Solve the MOP:

$$f^*_{\mathrm{Mom},d} = \inf\{\langle \Lambda, f\rangle \mid \Lambda \in \mathcal{L}_{2d}(\pm(h + s\varepsilon), r^2 - f),\ \langle \Lambda, 1\rangle = 1\};$$

(vii) Extract a minimizer η and check that $h(\xi)h(\eta) < 0$.

Output: False if the MOP is not feasible, true if the MOP is feasible and $h(\xi)h(\eta) < 0$.

In step (i) we fix a generic real projection such that X is birational to $\mathrm{cl}(\pi(X))$ (Lemma 4.4.5).
In steps (ii) and (iii) we compute a minimal degree polynomial h of the hypersurface $\mathrm{cl}(\pi(X))$, scaled so that one of its coefficients is 1 and stop if it has non-real coefficients.
In steps (iv), (v) and (vi) we check if the real polynomial h defines a real radical ideal, using Theorem 4.4.4. We find $\xi \in \mathbb{R}^{k+1}$ where h is not vanishing, and then search another point where h has opposite sign, by Moment Optimization.
If h does not change sign then $\mathcal{V}_{\mathbb{R}}(h + s\varepsilon) = \emptyset$ and the MOP will not be feasible (see for instance [LLR08]).
On the other hand if h changes sign there exist $\eta \in \mathbb{R}^{k+1}$ such that $h(\xi)h(\eta) < 0$. If $\|\eta - \xi\|_2 < r$ and $0 < \varepsilon \leq f(\eta)$ then the MOP has a solution. For generic ξ the minimizer will be a unique smooth point, the MOP will be exact (since we added the ball constraint $r^2 - f \geq 0$, the Archimedean property holds and generically the moment hierarchy is exact, see Corollary 3.5.9), and we can certify that h changes sign. The constraint $r^2 - \|\mathbf{x} - \xi\|_2^2 \geq 0$ is not necessary if $\mathcal{V}_{\mathbb{R}}(h)$ is compact, since in this case the Archimedean hypothesis is already satisfied.
The correctness of Algorithm 4.4.1 follows from Corollary 4.4.11.

4.4.4 Examples

We test Algorithm 4.4.1 for two simple cases, using the Julia packages `MomentTools.jl` and `MultivariateSeries.jl`.

Example 4.4.12. We check that the irreducible polynomial $h = x^2+y^2 \in \mathbb{R}[x,y]$ defines an ideal $I = (h)$ that is not real radical. We randomly choose $\xi = (-1.5667884102749219, -0.5028780359864093)$, where $h(\xi) > 0$. We check that h does not change sign, detecting the infeasibility of the optimization problem.

```
X = @polyvar x y
h = x^2 + y^2
s = sign(h(X => xi))
dist = sum((xi - vec(X)).^2)
e = 0.01
v, M = minimize(dist, [h+s*e], [9 - dist], X, 4, optimizer);
```

The termination status `termination_status(M.model)` of the optimization:

```
INFEASIBLE::TerminationStatusCode = 2
```

shows the infeasibility of the moment optimization program and that I is not real radical.

In the same way we detect the sign change. For $h = x^2 + y^2 - 1$ and ξ as above, we find $\eta = (-0.9473807839956285, -0.30408822493309284)$ and $h(\xi)h(\eta) < 0$.

In the previous examples we could avoid the ball constraint $r^2 - \|\mathbf{x} - \xi\|_2^2 \geq 0$, since in these cases $\mathcal{V}_{\mathbb{R}}(h)$ is compact and the Archimedean condition is already satisfied.

4.5 Computing the real radical

With the main ingredients, we can now describe the algorithm for computing the real radical of an ideal $I = (\mathbf{f})$, presented as the intersection of real prime ideals. The steps, summarised in Algorithm 4.5.1, are detailed hereafter.

In step (ii) we compute a generic element of $\mathcal{L}_{2d+2}(\pm\mathbf{f})$ solving a MOP with a constant objective function.

In step (iii) we use Algorithm 4.3.1 to compute the graded basis $\mathbf{k}$.

In step (iv) we find the irreducible components of the variety $\mathcal{V}_{\mathbb{C}}(\mathbf{k})$, described by witness sets (see Section 4.2.2). The embedded components of $(\mathbf{k})$ are not recovered by this technique.

In step (v) we control if the irreducible components of $\mathcal{V}_{\mathbb{C}}(\mathbf{k})$ are real, using Algorithm 4.4.1.

In step (vii), the equations defining X_i are obtained from $n+1$ generic projections. In particular, the equation of a generic projection of X_i used in step (ii) of Algorithm 4.4.1 provides one of the defining equation, say $h_{i,1}$.

We prove the correctness of the algorithm. By Theorem 4.2.1 we have $\mathcal{V}_{\mathbb{R}}(\mathbf{k}) = \mathcal{V}_{\mathbb{R}}(\mathbf{f})$ for $d \geq \max(\deg(\mathbf{f}))$. Let $\mathfrak{p}_i = (\mathbf{h}_i)$ in step (vii). By construction $\mathcal{V}_{\mathbb{R}}(\mathbf{k}) = \bigcup_i (X_i)_{\mathbb{R}} = \bigcup_i \mathcal{V}_{\mathbb{R}}(\mathfrak{p}_i) = \mathcal{V}_{\mathbb{R}}(\bigcap_i \mathfrak{p}_i)$. If step (v) succeeds, all the $\mathfrak{p}_i$'s are real radical, and thus $\bigcap_i \mathfrak{p}_i$ is real radical. Since $\mathcal{V}_{\mathbb{R}}(\mathbf{f}) = \mathcal{V}_{\mathbb{R}}(\bigcap_i \mathfrak{p}_i)$, by the Real Nullstellensatz $\bigcap_i \mathfrak{p}_i = \sqrt[\mathbb{R}]{\mathbf{f}}$ and the $\mathfrak{p}_i$ are the real prime ideal lying over $(\mathbf{f})$. The loop stops for some $d \gg 0$ by Theorem 4.2.1.

Algorithm 4.5.1 computes the minimal real prime ideals lying over $(\mathbf{f})$, but does not check that the equations $\mathbf{k}$ define a real radical ideal. If the ideal $(\mathbf{k})$ has no embedded component and the prime ideals $\mathfrak{p}_i$ are of multiplicity 1 (checked with the Jacobian criterion for $\mathbf{h}$ at a witness point of $\mathfrak{p}_i$), then the success of step (v) implies that $\mathbf{k} = \mathrm{Ann}_d(\Lambda^*)$ defines the real radical of $(\mathbf{f})$.

Algorithm 4.5.1: Real radical

Input: Polynomials $\mathbf{f} = (f_1, \dots, f_s) \subset \mathbb{R}[\mathbf{x}]$.
$d := \max(\deg(\mathbf{f}_i), i = 1, \dots, s) - 1$; success := false;
Repeat until success

(i) $d := d + 1$

(ii) Compute a generic element Λ^* of $\mathcal{L}_{2d+2}(\pm\mathbf{f})$

(iii) Compute a graded basis $\mathbf{k}$ of $\mathrm{Ann}_d(\Lambda^*)$ (Algorithm 4.3.1)

(iv) Compute the numerical irreducible components X_i of $V_{\mathbb{C}}(\mathbf{k})$ (described by witness sets)

(v) For each component X_i, check that X_i is real (Algorithm 4.4.1). If not repeat from step (i).

(vi) Set success := true

(vii) For each component X_i compute defining equations $\mathbf{h}_i = \{h_{i,1}, \dots, h_{i,n+1}\}$ of X_i

Output: The polynomials $\mathbf{h}_i$ generating the minimal real prime ideals $\mathfrak{p}_i$ lying over $(\mathbf{f})$.

Algorithm 4.5.1 can be simplified in the case where $\mathcal{V}_{\mathbb{R}}(\mathbf{f})$ is finite. We can check that $(\mathbf{k}) = \sqrt[\mathbb{R}]{\mathbf{f}}$, for $\mathbf{k} = \mathrm{Ann}_d(\Lambda^*)$, using the flat extension criterion. We can also detect this condition with the initial of $\mathbf{k}$, see **??** . In this case, Λ^* extends to a positive linear functional on $\mathbb{R}[\mathbf{x}]$ and $(\mathbf{k}) = \sqrt[\mathbb{R}]{\mathbf{f}}$.

Similarly, when the ideal $(\mathbf{k})$ is prime, one only needs to check that it is real (using Algorithm 4.4.1 on a generic projection), steps (iv), (vii) can be skipped and we obtain $(\mathbf{k}) = \sqrt[\mathbb{R}]{\mathbf{f}}$. When $(\mathbf{k})$ is real radical, the algorithm can even output directly $(\mathbf{k}) = \sqrt[\mathbb{R}]{\mathbf{f}}$.

4.6 Examples

We illustrate Algorithm 4.5.1, with the Julia package `MomentTools.jl`[1], using the semidefinite optimizer `Mosek`.

The isolated singular locus of a real surface

Example 4.6.1. Let $f = -10z^4 + x^3 - 3x^2z + 3xz^2 + 20yz^2 - z^3 - 10x^2 + 20xz - 10y^2 - 10z^2, g = 5 - (x^2 + y^2 + z^2)$ and $S = \{\xi \in \mathbb{R}^3 \mid f(\xi) = 0,\ g(\xi) \geq 0\}$. We want to compute the S-radical of $I = (f)$, which is equal to $(z - x, x^2 - y)$.

```
X = @polyvar x y z
f = -10*z^4 + x^3 - 3*x^2*z + 3*x*z^2 + 20*y*z^2
        - z^3 - 10*x^2 + 20*x*z - 10*y^2 - 10*z^2
g = 5 - (x^2+y^2+z^2)
v, M = minimize(one(f),[f], [g], X, 6, optimizer)
sigma = get_series(M)[1]
```

[1] `https://gitlab.inria.fr/AlgebraicGeometricModeling/MomentTools.jl`

```
L = monomials(X,0:3)
K,In,P,B = annihilator(sigma, L)
```

We compute a generic positive linear functional Λ (by optimising the constant function 1 on S), a graded basis K of $(\mathrm{Ann}_d(\Lambda))$, the initial monomials In of K, a basis P of $\frac{\mathbb{R}[\mathbf{x}]}{(\mathrm{Ann}_d(\Lambda))}$ orthogonal with respect to $\langle\cdot,\cdot\rangle_\Lambda$ and a monomial basis B of $\frac{\mathbb{R}[\mathbf{x}]}{(\mathrm{Ann}_d(\Lambda))}$. The elements of K are:

```
z - 0.999999935776211x - 2.027089868945844e-9y + 1.9280308682132505e-9
x² - 1.9114608711668615e-8x - 0.9999998601127081y - 2.6012502193917264e-7
```

These polynomials define a parametrisation of parabola and thus generate a real radical ideal. They are approximation of the generators of the S-radical of I within an error 3.e-7.

We can obtain the generators also using a slack variable s, and replacying the inequality $g \geq 0$ by the equation $g - s^2 = 0$. In this case the elements of K are:

```
z - 0.9999999987418964x - 2.0081938216111927e-9y + 1.848080975279204e-9
x² + 5.417748642831503e-10x - 0.9999999813624691y
     - 4.507056024417168e-23s - 2.369265117430075e-8
s²  + 2.531532655747432e-22ys - 7.729278487211091e-23xs
      - 2.0732509876020901e-22s + 0.9999999794170498y²
      + 1.1737503831818984e-8xy + 2.0000000080371674y
      - 1.4039307522382754e-8x - 4.999999978855321
```

and the generators of the S-radical are approximately $\mathsf{K} \cap \mathbb{R}[x,y,z]$.

Example 4.6.2. We compute equations for the hold of the Whitney umbrella. Let $f = x^2 - y^2 z, g = 1 - (x^2 + y^2 + (z+2)^2)$ and $S = \{\xi \in \mathbb{R}^3 \mid f(\xi) = 0,\ g(\xi \geq 0)\}$. We compute the S-radical of $I = (f)$, which is equal to (x,y). Proceding as above, we obtain for K, the polynomials:

```
x + 3.1388489268444904e-21,  y + 3.6567022687420305e-21
```

These polynomials are a good approximation of the generators (x,y) of the real radical, defining the singular locus of the Whitney umbrella.

Components of different dimensions

Example 4.6.3. This example is taken from [Ros09, ex. 9.6]. We want to compute the real radical of $I = (f_1, f_2, f_3) \subset \mathbb{R}[x,y,z]$, where:

$$\begin{aligned} f_1 &= x^2 + xy - xz - x - y + z \\ f_2 &= xy + 2y^2 - yz - x - 2y + z \\ f_3 &= xz + yz - z^2 - x - y + z. \end{aligned}$$

Its variety has three irreducible components, two lines and a point, defined by the real prime ideals $\mathfrak{p}_1 = (x-z, y)$, $\mathfrak{p}_2 = (x-z+1, y-1)$ and $\mathfrak{m} = (x-1, y-1, z-1)$. In the primary decomposition of I there is an embedded component $\mathfrak{m}'$, corresponding to the point $(1,0,1) \in \mathcal{V}(\mathfrak{p}_1)$ which has multiplicity two. The real radical of I is $\sqrt[\mathbb{R}]{I} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m} = (y^2 - y, x^2 - 2xz + z^2 + x - z, xz + yz - z^2 - x - y + z, xy + xz - z^2 - 2x - y + 2z)$.

We compute $\sqrt[\mathbb{R}]{I}$ as described in the algorithm.

```
v, M = minimize(one(f1),[f1,f2,f3], [], X, 8, optimizer)
sigma = get_series(M)[1]
L = monomials(X,0:3)
K,I,P,B = annihilator(sigma, L)
```

The elements of K are:

```
xz - 0.9999999985579915x² - 0.9999999940764733xy
     + 0.9999999838152133x + 0.9999999868597321y
     - 0.9999999838041349z - 2.550976860304921e-10
y² + 4.386341684978274e-7x² + 3.2135911001749273e-7xy
     - 8.511512801700947e-7x - 1.0000008530709377y
     + 9.888494964176088e-7z - 5.851033908621897e-8
yz + 8.763853490689755e-7x² - 0.9999993625797754xy
     + 0.9999983122334805x - 1.6948939787209127e-6y
     - 0.9999980367703514z - 1.1680315895740145e-7
z² - 0.9999991215344914x² - 1.99999935020258xy
     + 2.99999828318184x + 1.9999982828997438y
     - 2.999998007995895z - 1.1724998920381591e-7
```

which are approximately (within an error of 1.e-6) generators of $\sqrt[\mathbb{R}]{I}$.

Example 4.6.4. This example is taken from [BHL16, p. 8.2]. We want to compute the real radical of $I = (f_1, f_2, f_3) \subset \mathbb{R}[x,y,z]$, where:

$$
\begin{aligned}
f_1 &= xyz \\
f_2 &= z(x^2+y^2+z^2+y) \\
f_3 &= y(y+z).
\end{aligned}
$$

The associated complex variety has four irreducible components: two conjugates lines intersecting in the origin: $\mathfrak{p}_1 = (x - iz)$ and $\mathfrak{p}_2 = (x + iz)$, another line defined over $\mathbb{R}$ ($\mathfrak{p} = (y,z)$, double for **f**) and a point $\mathfrak{m} = (x, 2y+1, 2z-1)$. The real variety is given by the line $\mathfrak{p} = (y,z)$ and the point $\mathfrak{m} = (x, 2y+1, 2z-1)$. The real radical is $\sqrt[\mathbb{R}]{I} = \mathfrak{p} \cap \mathfrak{m} = (yx, z+y, y^2 + \frac{y}{2})$, since the intersection of two real radical ideals is real radical.

We first verify that **f** does not define a real radical ideal, using Algorithm 4.4.1. We compute a numerical irreducible decomposition, and find a witness set for a dimension 0, degree 1 component. Then we sample enough points, and we project using the generic real projection $\pi : \mathbb{C}^3 \to \mathbb{C}^2$ given by $\begin{pmatrix} 0.707973 & 0.95564 & 0.821304 \\ 0.814441 & 0.474915 & 0.467363 \end{pmatrix}$, to reduce to the hypersurface case. The projected sampled points are:

```
[0.18832975413312392-2.3958515346371136i,-0.5349465754402629-2.0112458147560544i]
[0.15806516802973358-2.271846451408753i,-0.5240184003247248-1.9012487898978656i]
[0.1989396484844097-2.436629789668384i,-0.5380030719003553-2.0476066622156854i]
```

and, using coordinates s,t on $\mathbb{C}^2$, we obtain as equation for the hypersurface:

$$h = (0.9999999999999999 + 0.0i)s + (-1.0892629121438033 + 0.38335802271492037i)t$$

Since h is not a real polynomial, then **f** does not define a real radical ideal.

We now compute $\sqrt[\mathbb{R}]{I}$ as described in Algorithm 4.5.1 and obtain for K:

```
z - 6.53338688785662e-19x + 0.9995827809845268y - 0.00020850768649473272
xy - 1.4685109255649737e-19x² + 5.9730164512226755e-6x
     + 2.1320912413237275e-19y + 1.0655056374451632e-19
y² - 2.268705086623265e-6x² + 1.88498770272315e-19x
     + 0.4998194337295852y + 4.384653173789382e-6
```

approximating (within an error of `5.e-4`) the generators of $\sqrt[\mathbb{R}]{I}$.

Remark. It is interesting going into more details of the last example to analyze the stability of our algorithm with respect to numerical errors.

The generators K of the real radical have been computed within an error of `5.e-4`. This error is relatively big, and it is due to the low precision of computation of the generic positive linear functional Λ^*. If one computes directly with `Bertini` a numerical irreducible decomposition of $\mathcal{V}_{\mathbb{C}}(\mathsf{K})$, we obtain three zero dimensional components, instead of one point and one line. This happens because `Bertini` requires an input with higher precision. However, we can detect the problem, since two of these three points are very close, suggesting that they should collapse into a line. However, if we round the polynomials in K at order `1.e-4` we obtain the correct irreducible decomposition, and we can proceed with the algorithm, as follows.

The zero dimensional component is clearly a real point, and it is not necessary to apply Algorithm 4.4.1.

Let's turn our attention to the one dimensional component. The projected sampled points are:

```
[-0.17334774340829084+0.15549627481778386i,0.08061255672349628-0.07231102077006586i]
[-0.10989849893673652+0.044164323338748625i,0.05110651459995326-0.020537902313007134i]
[-0.15061603261213993-0.42091242552217645i,0.0700415432799536+0.19573849714393235i]
```

and, using coordinates s,t on $\mathbb{C}^2$, we obtain as equation for the hypersurface:

$$h = s + (2.1503814102172614 + 3.9313021464326093e - 16i)t.$$

This is approximately a real equation, and we can verify using Algorithm 4.5.1 that it defines a real hypersurface. Then the one dimensional component is totally real, and finally since the one dimensional component does not contain the isolated point, we can conclude that K generates the real radical.

Example 4.6.4 describes a very complicated geometry, and it is a natural situation when numerical issues can arise. We propose in the next section possible systematic solutions to this challenging problem.

4.7 Limitations and perspectives

Algorithm 4.5.1 is a symbolic-numeric algorithm, whose output depends on the quality of the numerical tools that are involved. In particular, the numerical quality of the generic positive linear functional Λ^*, produced by a SDP solver, impacts the computation of generators of the real radical. This computation depends on a threshold used to determine when a polynomial is in the annihilator.

A natural solution to numerical issues is the use of high-precision semidefinite solvers, such as `SDPA-GMP`. In this way, the quality of approximation of the generic positive linear functional will be improved and the computation of the basis K will be more accurate. Another possibility could be an extra rounding or projection procedure, based on the semidefinite description of the real radical, to improve the quality of K.

Another improvement of the algorithm (in particular, of Algorithm 4.3.1), is the exploitation of the sparsity structure. Indeed, the algorithm is presented and implemented in the dense case, while most concrete problems (and also our examples) are sparse. The use of the sparsity structure would lead to two advantages:

- the improvement of the performance of the algorithm;
- the possibility to obtain generic Λ^* with more accuracy.

From a different point of view, a future perspective could be to exploit geometrical and algebraic properties of real systems of equations to create challenging instances of semidefinite programs, and test the quality of different solvers on extracting a positive generic Λ^*.

Conclusion and Perspectives

In this thesis, we have investigated exact and approximate representation properties of positive polynomials and truncated linear functionals, motivated from problems in polynomial optimization.

On the polynomial side, in Chapter 2 we investigated exact, effective representations of strictly positive polynomials on basic closed semialgebraic sets as elements of the quadratic module, under the Archimedean assumption: that is, the *effective Putinar's Positivstellensatz*. Our result improves significantly the previous bound in the literature, and have several consequences.

Indeed, we can interpret the effective Putinar's Positivstellensatz as a result of quantitative approximation of positive polynomials, and then deduce the first general polynomial convergence for Lasserre's hierarchies. On the dual side, we deduce convergence rates of truncated positive linear functionals (or truncated pseudo-moment sequences) to measures.

Moreover, both the techniques of the proof (a combination of approximation theory and semialgebraic geometry) and the final result itself open several interesting questions and perspectives, see Section 2.6.

We then moved to exact representations of positive truncated linear functionals. In Chapter 3 we investigated properties of the dual cones of truncated quadratic modules, and we introduced the concept of *exactness* for Lasserre's moment hierarchy, that is closely related to the *flat truncation* property. We used this description and the zero dimensionality condition to study flat truncation in polynomial optimization. We gave the first necessary and sufficient condition for flat truncation, under the finite convergence assumption. As corollaries, we concluded that flat truncation holds if the generic Boundary Hessian Condition holds at every minimizer of the objective function on the semialgebraic set, and we gave a unified presentation of different results in the zero dimensional case.

The theory developed and the results obtained lead naturally to possible further investigations, described in Section 3.7.

As an application, we present a new algorithm to compute the real radical of an ideal I. Our algorithm, described in Chapter 4, makes use of several ingredients: properties of truncated positive linear functionals, the quotient structure $\mathbb{R}[\mathbf{x}]/I$, numerical algebraic geometry, and effective criterions in real algebraic geometry, verified solving a hierarchy of semidefinite programs. The algorithm is devoted to the challenging positive dimensional case, and can be used to find equations for the real irreducible components and the generators of the real radical.

Possible future improvements of the algorithm are discussed in Section 4.7.

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